Resolvent conditions for the control of parabolic equations

Thomas Duyckaerts, Luc Miller

RESOLVENT CONDITIONS FOR THE CONTROL OF PARABOLIC EQUATIONS

THOMAS DUYCKAERTS[1] AND LUC MILLER[2]

ABSTRACT. Since the seminal work of Russell and Weiss in 1994, resolvent conditions for various notions of admissibility, observability and controllability, and for various notions of linear evolution equations have been investigated intensively, sometimes under the name of infinite-dimensional Hautus test. This paper sets out resolvent conditions for null-controllability in arbitrary time: necessary for general semigroups, sufficient for analytic normal semigroups.

For a positive self-adjoint operator A, it gives a sufficient condition for the null-controllability of the semigroup generated by $-A$ which is only logarithmically stronger than the usual condition for the unitary group generated by iA. This condition is sharp when the observation operator is bounded. The proof combines the so-called "control transmutation method" and a new version of the "direct Lebeau-Robbiano strategy". The improvement of this strategy also yields interior null-controllability of new logarithmic anomalous diffusions.

1. Introduction

This section describes briefly the control property under investigation in the semigroup framework (we refer to the monograph [TW09] for a full account), some previous results on resolvent conditions, the purpose, main results and plan of this paper, and some applications to distributed parameter systems.

1.1. **Preliminaries on control systems.** Let $-A$ be the generator of a strongly continuous semigroup on a Hilbert space $\mathcal{E}$ and C be a bounded operator from the domain $\mathrm{D}(A)$ with the graph norm to another Hilbert space $\mathcal{F}$. Now $\|\cdot\|$ denotes the norms in $\mathcal{E}$ and $\mathcal{F}$ and also the associated operator norms.

Recall the usual *admissibility* condition (for some time $T > 0$ hence all $T > 0$),

$$\exists\, \mathrm{K}_T > 0,\ \forall v \in \mathrm{D}(A), \quad \int_0^T \|Ce^{-tA}v\|^2 dt \leqslant \mathrm{K}_T \|v\|^2, \tag{1}$$

which implies that the output map $v \mapsto Ce^{-tA}v$ from $\mathrm{D}(A)$ to $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{F})$ has a continuous extension to $\mathcal{E}$ (n.b. the optimal admissibility constant $T \mapsto \mathrm{K}_T$ is nondecreasing). E.g. (1) holds when C is bounded on $\mathcal{E}$.

If the admissibility condition (1) holds, then null-controllability at time T (the definition is not needed here) is equivalent to *final-observability* at time T, i.e.

$$\exists \kappa_T > 0,\ \forall v \in \mathrm{D}(A), \quad \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt. \tag{2}$$

The control property investigated in this paper is (2) *for all* $T > 0$.

---

[1]LAGA, UMR 7539, Institut Galilée, Université Paris 13, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

[2]MODAL'X, EA 3454, Bât. G, Université Paris Ouest Nanterre La Défense, 200 Av. de la République, 92001 Nanterre Cedex, France.

2010 *Mathematics Subject Classification.* 93B05, 93B07, 35K90, 35Q93.

Keywords: Null controllability; Hautus test; resolvent estimates; linear parabolic equations.

[2]*Corresponding author. E-mail address:* Luc.Miller@math.cnrs.fr .

Here C is interpreted as an *observation operator* and (2) as the continuous prediction of the final state by observing the evolution between initial and final times.

Recall that κ_T is the *control cost*: it is the ratio of the size of the input over the size of the initial state which the input steers to the zero final state in a lapse of time T. It blows up as T tends to zero, e.g. like $\exp(1/T)$ for the heat equation, cf. [Sei08]. We refer to [Sei05, Mil10a] for a more extensive presentation.

Remark 1.1. When a multiple of the identity operator is added to A these notions do not change, the constants K_T and κ_T change but not their asymptotics as $T \to 0$, e.g. the value of $\limsup_{T\to 0} T^\beta \ln \kappa_T$ for $\beta > 0$ does not change. For this reason in some of our statements we may assume without real loss of generality that the semigroup generated by $-A$ is bounded or even exponentially stable.

1.2. **Background on resolvent conditions for observability.** The following resolvent condition was introduced in [RW94]: $\exists M > 0$,

$$\|v\|^2 \leqslant \frac{M}{(\operatorname{Re}\lambda)^2}\|(A-\lambda)v\|^2 + \frac{M}{\operatorname{Re}\lambda}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0. \tag{3}$$

N.b. from now on, such an equation means: $\exists M > 0$, $\forall v \in \mathrm{D}(A)$, $\forall \lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$, the inequality in (3) holds. Russell and Weiss proved that (3) is a necessary condition for exact observability in infinite time of exponentially stable semigroups. Their conjecture that it is sufficient was disproved in [JZ04].

When the generator is skew-adjoint (equivalently when the semigroup is a unitary group) a similar resolvent condition is both necessary and sufficient for exact observability in finite time, cf. [Mil12, theorem 3.8] and [Mil12, proposition 2.9(iii)]):

Theorem 1.2. *Let $\mathcal{A}$ be a self-adjoint operator on a Hilbert space $\mathcal{W}$ and $\mathcal{C}$ be a bounded operator from $\mathrm{D}(\mathcal{A})$ with the graph norm to another Hilbert space.*

Assume the usual admissibility condition (for some time $\tau > 0$ hence all $\tau > 0$),

$$\exists\, \mathrm{Adm}_\tau > 0,\ \forall v \in \mathrm{D}(\mathcal{A}), \quad \int_0^\tau \|\mathcal{C}e^{it\mathcal{A}}v\|^2 dt \leqslant \mathrm{Adm}_\tau \|v\|^2. \tag{4}$$

The resolvent condition: $\exists M > 0$, $\exists m > 0$,

$$\|v\|^2 \leqslant M\|(\mathcal{A}-\lambda)v\|^2 + m\|\mathcal{C}v\|^2, \quad v \in \mathrm{D}(\mathcal{A}), \quad \lambda \in \mathbb{R}, \tag{5}$$

is equivalent to exact observability in some time $\tau > 0$, i.e. $\exists \tau > 0$,

$$\exists\, \mathrm{Obs}_\tau > 0,\ \forall v \in \mathrm{D}(\mathcal{A}), \quad \|v\|^2 \leqslant \mathrm{Obs}_\tau \int_0^\tau \|\mathcal{C}e^{it\mathcal{A}}v\|^2 dt. \tag{6}$$

More precisely, (5) *implies* (6) *for all $\tau > \pi\sqrt{M}$ with $\mathrm{Obs}_\tau \leqslant \frac{2m\tau}{\tau^2 - \pi^2 M}$.*

Moreover, $\mathbb{R}$ in (5) *may be replaced equivalently by the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ with the same constants M and m. E.g. if $\mathcal{A}$ is positive self-adjoint then $\lambda \in \mathbb{R}$ in* (5) *may also be replaced by $\lambda > 0$ (more generally by $\lambda \geqslant \inf \mathcal{A} := \inf \sigma(\mathcal{A})$).*

We refer to [Liu97, ZY97, BZ04, Mil05] and to [Mil12] for more background and references. This result was extended to more general groups in [JZ09, theorem 1.2]. Resolvent conditions equivalent to (4) were introduced in [Erv09, Erv11] and generalized in [Mil12]. We refer to [EZZ08, Erv09, Erv11, Mil12] for applications to discretization.

This paper addresses resolvent conditions for the null-controllability (2) of heat-like semigroups, i.e. when A is positive self-adjoint or more generally when the semigroup is normal analytic (the definition can be found at the beginning of theorem 3.7 and of §3). Resolvent conditions for the weaker notion of final-observability in infinite time:

$$\exists T > 0, \exists \kappa_T > 0, \forall v \in \mathcal{E}, \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^\infty \|Ce^{-tA}v\|^2 dt, \tag{7}$$

was also investigated in [JZ09] for exponentially stable normal semigroups (in this framework (7) implies (2) for *some* time T), cf. remark 3.8. But it seems that resolvent conditions for final-observability for *any* $T > 0$ in (2) has not been previously investigated, although it is a very natural notion for heat-like semigroups.

An other condition called the (α, β) Hautus test is introduced in [JS07, definition 3.5]. Other related papers are [JP06, JPP07, JPP09, XLY08].

1.3. **Main results.** For simplicity, we focus in this introduction on the case where the observation operator C is bounded from $\mathcal{E}$ to $\mathcal{F}$. We refer to later sections for the full statement of the main theorems under more general admissibility conditions.

We first state a sufficient resolvent condition for final-observability for any $T > 0$ (cf. theorems 4.3 and 4.5).

Theorem 1. *Assume A is positive self-adjoint and C is bounded (from $\mathcal{E}$ to $\mathcal{F}$).*

If the resolvent condition with logarithmic factor *: $\exists M_* > 0$,*

$$\|v\|^2 \leqslant \frac{M_* \lambda}{(\log(\lambda+1))^\alpha} \left(\frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathrm{D}(A), \quad \lambda > 0, \tag{8}$$

holds for some power $\alpha > 2$, then final-observability (2) *for the semigroup generated by $-A$ holds for any time $T > 0$.*

If the resolvent condition with power-law factor *: $\exists M_* > 0$,*

$$\|v\|^2 \leqslant M_* \lambda^\delta \left(\frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathrm{D}(A), \quad \lambda > 0, \tag{9}$$

holds for some power $\delta \in [0,1)$, then final-observability (2) *for the semigroup generated by $-A$ holds for any time $T > 0$ with the control cost estimate $\kappa_T \leqslant c e^{c/T^\beta}$ for $\beta = \frac{1+\delta}{1-\delta}$ and some $c > 0$.*

N.b. in the family of resolvent conditions (9), $\delta = 0$ is equivalent to the exact observability of the corresponding wave equation $\ddot{w} + Aw = 0$, $\delta = 1$ is implied by the exact observability of the corresponding Schrödinger equation $\dot{\psi} - iA\psi = 0$, $\delta = -1$ is the condition (3) of Russell and Weiss restricted to real λ.

Combining theorem 1 with theorem 1.2 yields the next statement. In a nutshell, it says that the observability of the Schrödinger equation $\dot{\psi} - i\mathcal{A}\psi = 0$ with a self-adjoint differential operator $\mathcal{A}$ implies the observability of some heat equations $\dot{\phi} + A\phi = 0$ with A of "higher order" than $\mathcal{A}$. The proof is completed by the convexity theorem [Mil12, Theorem 3.2]: (5) implies (9) for $A = \mathcal{A}^\gamma$ and $\delta = \frac{2}{\gamma} - 1$, and (8) for $A = \mathcal{A} \log^{\alpha/2}(1+\mathcal{A})$, cf. [Mil12, Example 3.4].

Corollary 2. *Assume $\mathcal{A}$ is positive self-adjoint and $\mathcal{C} = C$ is bounded, and consider*

$$A = \mathcal{A}^\gamma, \ \gamma > 1, \qquad \textit{or} \qquad A = \mathcal{A} \log^{\alpha/2}(1+\mathcal{A}), \ \alpha > 2.$$

If exact observability (6) *for the Schrödinger group $(e^{it\mathcal{A}})_{t\in\mathbb{R}}$ holds for some time, then final-observability* (2) *for the heat semigroups $(e^{-tA})_{t\geqslant 0}$ holds for any time.*

The condition $\gamma > 1$ in corollary 2 is sharp by the following example of the harmonic oscillator observed from a half line (cf. proposition 5.1).

Proposition 3. *Consider the positive self-adjoint operator $A = -\partial_x^2 + x^2$ on the space $\mathcal{E} = L^2(\mathbb{R})$ of square-summable functions on $\mathbb{R}$. Define the bounded operator C on $\mathcal{E} = \mathcal{F}$ as the multiplication by the characteristic function of $(-\infty, x_0)$, $x_0 \in \mathbb{R}$.*

Exact observability for the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$ holds for some time, but final-observability (2) *for the heat semigroup $(e^{-tA})_{t\geqslant 0}$ does not hold for any time.*

This example also proves that condition $\delta < 1$ in theorem 1 is sharp (since (9) holds with $\delta = 1$ by theorem 1.2). Another example given in §5.2 strongly suggests that condition $\alpha > 2$ in theorem 1 is also sharp (cf. the second paragraph of §5).

As opposed to the resolvent condition for unitary groups in theorem 1.2, the sufficient condition (9) is not necessary. Instead of this resolvent condition with power-law factor, the examples in remark 2.6 lead us to rather consider the following *resolvent conditions with exponential factor* with some powers $\alpha > 0$: $\exists a > 0$,

$$\|v\|^2 \leqslant a e^{a(\operatorname{Re}\lambda)^\alpha} \left(\|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0. \tag{10}$$

Indeed we prove that they are necessary for final-observability (cf. theorem 2.4).

Theorem 4. *If* (1) *and* (2) *hold for some T then* (10) *holds with power $\alpha = 1$. If* (2) *holds moreover for all $T \in (0, T_0]$ with the control cost $\kappa_T = c e^{c/T^\beta}$ for some $\beta > 0$, $c > 0$, $T_0 > 0$, then* (10) *holds with power $\alpha = \frac{\beta}{\beta+1} < 1$.*

In §3, we give an exponential resolvent condition stronger than (10) which is sufficient for final-observability (cf. theorem 3.7). For proving this and the other sufficient condition (9) in theorem 1, we use the Lebeau-Robbiano strategy initiated in [LR95] as revisited in [Mil10a] (cf. also [Sei08, TT10]). Since this version of the strategy falls short of proving the weaker logarithmic sufficient condition (8), we stretched it to this purpose by dropping the requirement that it should provide an explicit estimate of the control cost κ_T as $T \to 0$.

This new version of the direct Lebeau-Robbiano strategy of [Mil10a] is presented in §6 for its own sake. In particular, it yields the following logarithmic improvement. For simplicity, we state it for normal semigroups, although it is valid within the more general framework of [Mil10a], cf. theorems 3.5 and 6.1. Definitions of normal semigroups and spectral subspaces $\mathbf{1}_{\operatorname{Re} A<\lambda}\,\mathcal{E}$ can be found at the beginning of §3. E.g. if A is a positive self-adjoint operator then $(e^{-tA})_{t\geqslant 0}$ is a normal semigroup.

Theorem 5. *Assume the admissibility condition* (1) *and that $-A$ generates a normal semigroup. If the* logarithmic observability condition on spectral subspaces

$$\|v\|^2 \leqslant a e^{a\lambda/((\log(\log\lambda))^\alpha \log\lambda)} \|Cv\|^2, \quad v \in \mathbf{1}_{\operatorname{Re} A<\lambda}\,\mathcal{E}, \quad \lambda > e. \tag{11}$$

holds for some $\alpha > 2$ and $a > 0$, then final-observability (2) *holds for all $T > 0$.*

The corresponding condition for the strategy of [Mil10a] was

$$\|v\|^2 \leqslant a e^{2a\lambda^\alpha} \|Cv\|^2, \quad v \in \mathbf{1}_{\operatorname{Re} A<\lambda}\,\mathcal{E}, \quad \lambda \geqslant \lambda_0, \tag{12}$$

for some $\alpha \in (0,1)$, $a > 0$ and $\lambda_0 > 0$ (n.b. this condition is sufficient for the same range of exponents $\alpha \in (0,1)$ as the second necessary condition of theorem 4). The term λ^α, $\alpha \in (0,1)$, in this earlier condition is replaced by $\lambda/\varphi(\lambda)$, with $\varphi(\lambda) = (\log(\log\lambda))^\alpha \log\lambda$ and $\alpha > 2$, in the new condition (11) of theorem 5. The condition (11) may be replaced by the weaker time-dependent condition parallel to the original in [LR95] (cf. (66) in theorem 6.1):

$$\|e^{-TA}v\|^2 \leqslant \frac{a}{T^a} e^{\frac{a\lambda}{\varphi(\lambda)}} \int_0^T \|Ce^{-tA}v\|^2 dt, \; v \in \mathbf{1}_{\operatorname{Re} A<\lambda}\,\mathcal{E}, \; T \in (0, T_0), \; \lambda \geqslant \lambda_0. \tag{13}$$

1.4. **Applications of the main results to PDEs.** Theorem 1 applies to *diffusions in a potential well* in the following way. Consider $A = -\Delta + V$ on $\mathcal{E} = L^2(\mathbb{R})$ with potential $V(x) = x^{2k}$, $k \in \mathbb{N}^*$. It is positive self-adjoint with domain $\mathrm{D}(A) = \left\{ u \in H^2(\mathbb{R}) \mid Vu \in L^2(\mathbb{R}) \right\}$. Let $C : \mathcal{E} \to \mathcal{F} = \mathcal{E}$ be the multiplication by the characteristic function $\chi_{(-\infty,x_0)}$ of a half line $(-\infty, x_0)$, $x_0 \in \mathbb{R}$. It is proved in [Mil10b] that they satisfy this power-law resolvent condition:

$$\|v\|^2 \leqslant \lambda^{1/k} \frac{M}{\lambda} \|(A-\lambda)v\|^2 + m\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda > 0,$$

and the decay of the first coefficient cannot be improved (due to a basic quasimode). Thus theorem 1 gives an alternative proof of [Mil08, theorem 1.10] in dimension one (n.b. in higher dimensions, theorem 1 also applies, but under a geometric condition on cones which is stronger than in [Mil08, theorem 1.10]):

Theorem 6. *The diffusion in the potential well $V(x) = x^{2k}$, $k \in \mathbb{N}$, $k > 1$,*

$$\partial_t \phi - \partial_x^2 \phi - V\phi = \chi_{(-\infty,x_0)} u, \quad \phi(0) = \phi_0 \in L^2(\mathbb{R}),\ u \in L^2([0,T]\times\mathbb{R}),$$

is null-controllable in any time, i.e. $\forall T > 0$, $\forall \phi_0$, $\exists u$ such that $\phi(T) = 0$.

Theorem 5 applies to *logarithmic anomalous diffusions* in the following way (cf. the more general theorem 6.3). Such anomalous diffusions consist in replacing the Laplacian in the usual heat equation by some functions of the Laplacian defined by the functional calculus of self-adjoint operators. The key PDE result is an interior observability estimate for sums of eigenfunctions of the Dirichlet Laplacian Δ proved in joint papers of Lebeau with Jerison and Zuazua using the boundary Carleman estimates due to Robbiano as improved in [LR95], cf. [JL99, LZ98, LRL09]. This milestone estimate writes as (12) with exponent $\alpha = 1/2$, i.e.

$$\|v\|^2 \leqslant c e^{c\sqrt{\lambda}} \|Cv\|^2, \quad v \in \mathbf{1}_{-\Delta<\lambda}\,\mathcal{E}, \quad \lambda > 0, \tag{14}$$

for some $c > 0$, where $\mathcal{E} = L^2(M)$, $C : \mathcal{E} \to \mathcal{F} = \mathcal{E}$ is the multiplication by the characteristic function χ_Ω of an open subset $\Omega \neq \emptyset$ of M, i.e. it truncates the input function outside the control region Ω. N.b. (14) is indeed an estimate of sums of eigenfunctions since the spectral space $\mathbf{1}_{-\Delta<\lambda}\,\mathcal{E}$ is just the linear span of the eigenfunctions of $-\Delta$ with eigenvalues lower than λ. We deduce from (14) that the logarithmic observability condition is satisfied by $A = -\sqrt{-\Delta}\varphi(\sqrt{-\Delta})$ for each of the following functions φ defined on $(0,+\infty)$, hence on the spectrum of $\sqrt{-\Delta}$,

$$\begin{aligned} &\varphi(\lambda) = (\log(1+\log(1+\lambda)))^\alpha \log(1+\lambda), \quad \alpha > 2, \\ \text{or} \quad &\varphi(\lambda) = (\log(1+\lambda))^\alpha, \quad \alpha > 1, \end{aligned} \tag{15}$$

(this computation is sketched before theorem 6.3). Thus theorem 5 proves

Theorem 7. *Consider the anomalous diffusion in the smooth connected bounded domain M of $\mathbb{R}^d$, defined by the Dirichlet Laplacian Δ and (15), with input u:*

$$\partial_t \phi + \sqrt{-\Delta}\varphi(\sqrt{-\Delta})\phi = \chi_\Omega u, \quad \phi(0) = \phi_0 \in L^2(M),\ u \in L^2([0,T]\times M).$$

It is null-controllable from any non-empty open subset Ω of M in any time $T > 0$.

For $\varphi(\lambda) = \lambda^\alpha$, this problem was first discussed in [MZ06] (basically for a one dimensional input u depending only on time not space, and $\alpha \in (-1,0)$), then null-controllability was proved for $\alpha > 0$ in [Mil06b], and the estimate of the control cost was improved into $\kappa_T = ce^{c/T^{1/\alpha}}$ for some $c > 0$, in [Mil10a, theorem 4.1]. It is still an open problem wether theorem 7 holds for $\varphi(\lambda) = 1$, cf. remark 6.4.

1.5. **Outline of the paper.** In the general framework of §1.1, §2 proves that some exponential resolvent conditions (10) are indeed necessary for null-controllability. It mainly consists in changing i into -1 in [Mil05, lemma 5.2]. It is also close to the proof in [RW94, theorem 1.2] that the stronger assumption of exact observability in infinite time implies (3). Parallel exponential resolvent conditions are proved necessary for admissibility (1).

In §3, some exponential resolvent conditions stronger than (10) are proved sufficient for null-controllability when A is normal. The proof is based on the *direct Lebeau-Robbiano strategy* of [Mil10a] for proving final-observability (cf. theorem 3.3), which is an outgrowth of the heat control strategy devised in [LR95].

When A is positive selfadjoint, §4 proves that null-controllability is implied by power-law resolvent conditions weaker than (3) and, thanks to §6, by even weaker logarithmic resolvent conditions. The proof combines the *direct Lebeau-Robbiano strategy* of [Mil10a] and the *control transmutation method* of [Mil06a]. N.b. this method uses an integral representation similar to Phung's in [Phu01, Phu02] to deduce the final-observability of the heat equation $\dot{v} + Av = 0$, with an explicit estimate of the fast control cost, from the exact observability of the wave equation $\ddot{w} + Aw = 0$ in some given time (cf. theorem 4.2).

Examples assessing the sharpness of these sufficient conditions are given in §5.

The independent §6 improves the Lebeau-Robbiano strategy in [Mil10a] when estimating the costs is not a goal, and applies it to logarithmic anomalous diffusions.

2. Necessary resolvent conditions for semigroups

The framework of this section is as in §1.1: $-A$ is the generator of a strongly continuous semigroup on $\mathcal{E}$ and $C \in \mathcal{L}(\mathrm{D}(A), \mathcal{F})$. This section examines which resolvent conditions are implied by null-controllability in time T, and similarly by admissibility. We first prove an auxiliary lemma.

Lemma 2.1. *For all* $T > 0$, $v \in \mathrm{D}(A)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$*:*

$$\frac{1 - e^{-2T\operatorname{Re}\lambda}}{4\operatorname{Re}\lambda}\|Cv\|^2 \leqslant \int_0^T \|Ce^{-tA}v\|^2 dt + \frac{1}{(\operatorname{Re}\lambda)^2}\int_0^T \|Ce^{-tA}(A-\lambda)v\|^2 dt, \tag{16}$$

$$\int_0^T \|Ce^{-tA}v\|^2 dt \leqslant \frac{1}{\operatorname{Re}\lambda}\|Cv\|^2 + \frac{2}{(\operatorname{Re}\lambda)^2}\int_0^T \|Ce^{-tA}(A-\lambda)v\|^2 dt. \tag{17}$$

Proof. The following definitions shall be convenient:

$$x(t) = e^{-tA}v, \quad z(t) = x(t) - e^{-t\lambda}v \quad \text{and} \quad f = (A-\lambda)v. \tag{18}$$

Since $-\dot{x}(t) = Ax(t) = e^{-tA}Av = e^{-tA}(\lambda v + f) = \lambda x(t) + e^{-tA}f$, we obtain $\dot{z}(t) = \dot{x}(t) + \lambda e^{-t\lambda}v = -\lambda z(t) - e^{-tA}f$ and therefore $z(t) = -\int_0^t e^{-(t-s)\lambda}e^{-sA}f\,ds$. Hence, by Cauchy-Schwarz inequality, $\|Cz(t)\|^2 \leqslant I_t \int_0^t e^{-(t-s)\operatorname{Re}\lambda}\|Ce^{-sA}f\|^2\,ds$ with $I_t = \int_0^t e^{-(t-s)\operatorname{Re}\lambda}\,ds = \int_0^t e^{-s\operatorname{Re}\lambda}\,ds \leqslant 1/\operatorname{Re}\lambda$. Fubini's theorem yields

$$\int_0^T \|Cz(t)\|^2 dt \leqslant I_T \int_0^T I_{T-s}\|Ce^{-sA}f\|^2 ds \leqslant \frac{1}{(\operatorname{Re}\lambda)^2}\int_0^T \|Ce^{-sA}f\|^2 ds.$$

Respectively plugging $e^{-t\lambda}v = x(t) - z(t)$ and $x(t) = e^{-t\lambda}v + z(t)$, we now have the following estimates which yield (16) and (17)

$$\int_0^T \|Ce^{-t\operatorname{Re}\lambda}v\|^2 dt \leqslant 2\int_0^T \|Cx(t)\|^2 dt + \frac{2}{(\operatorname{Re}\lambda)^2}\int_0^T \|Ce^{-sA}f\|^2\,ds,$$

$$\int_0^T \|Cx(t)\|^2 dt \leqslant 2\int_0^T e^{-2t\operatorname{Re}\lambda}dt\|Cv\|^2 + \frac{2}{(\operatorname{Re}\lambda)^2}\int_0^T \|Ce^{-sA}f\|^2\,ds.$$

□

A direct consequence of (16) is a necessary resolvent condition for admissibility:

Proposition 2.2. *The admissibility condition* (1) *implies the admissibility resolvent condition*

$$\|Cv\|^2 \leqslant L(\lambda)\|(A-\lambda)v\|^2 + l(\lambda)\|v\|^2, \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0, \tag{19}$$

with $L(\lambda) = \frac{4\,\mathrm{K}_T}{(1-e^{-2T\operatorname{Re}\lambda})\operatorname{Re}\lambda}$ *and* $l(\lambda) = \frac{4\,\mathrm{K}_T\operatorname{Re}\lambda}{(1-e^{-2T\operatorname{Re}\lambda})}$.

We now state necessary resolvent conditions for final-observability.

Proposition 2.3. *Let $B_T = \sup_{t\in[0,T]} \|e^{-tA}\|$ be the semigroup bound up to time T. If* (1) *and* (2) *hold then :* $\forall v \in \mathrm{D}(A)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$,

$$\|v\|^2 \leqslant 2e^{2T\operatorname{Re}\lambda}\left((B_T^2 + 2\kappa_T \mathrm{K}_T)\frac{\|(A-\lambda)v\|^2}{(\operatorname{Re}\lambda)^2} + \kappa_T \frac{\|Cv\|^2}{\operatorname{Re}\lambda}\right). \tag{20}$$

If moreover (2) *holds for all* $T \in (0, T_0]$ *with* $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$, $c_0, c, \beta > 0$, *then*

$$\|v\|^2 \leqslant a_0 e^{2a(\operatorname{Re}\lambda)^\alpha}\left(\frac{\|(A-\lambda)v\|^2}{(\operatorname{Re}\lambda)^2} + \frac{\|Cv\|^2}{\operatorname{Re}\lambda}\right), \quad v \in \mathrm{D}(A), \quad \operatorname{Re}\lambda > 0, \tag{21}$$

with $\alpha = \frac{\beta}{\beta+1}$, $a = c^{\frac{1}{\beta+1}}\frac{\beta+1}{\beta^\alpha}$ *and* $a_0 = 2(B_{T_0}^2 + c_0(1 + 2\,\mathrm{K}_{T_0}))\exp\frac{2c(\beta+1)}{T_0^\beta}$.

Proof. We keep the notations (18) of the proof of lemma 2.1. The semigroup bound yields $\|z(T)\| \leqslant I_T B_T \|f\| \leqslant \frac{B_T}{\operatorname{Re}\lambda}\|f\|$. Together with (2), it implies the following estimate of $v = e^{T\lambda}(x(T) - z(T))$:

$$\|v\|^2 \leqslant 2e^{2T\operatorname{Re}\lambda}\left(\kappa_T \int_0^T \|Cx(t)\|^2 dt + \frac{B_T^2}{(\operatorname{Re}\lambda)^2}\|f\|^2\right).$$

Plugging in the following consequence of (17) and (1)

$$\int_0^T \|Cx(t)\|^2 dt \leqslant \frac{1}{\operatorname{Re}\lambda}\|Cv\|^2 + \frac{2\,\mathrm{K}_T}{(\operatorname{Re}\lambda)^2}\|f\|^2,$$

completes the proof of (20).

Plugging $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$ in (20) and using that $T \leqslant T_0$ and $\mathrm{K}_T \leqslant \mathrm{K}_{T_0}$ yields:

$$\|v\|^2 \leqslant 2(c_0 + B_{T_0}^2 + 2c_0\,\mathrm{K}_{T_0})e^{2h_\lambda(T)}\left(\frac{\|Cv\|^2}{\operatorname{Re}\lambda} + \frac{\|(A-\lambda)v\|^2}{(\operatorname{Re}\lambda)^2}\right),$$

with $h_\lambda(T) = T\operatorname{Re}\lambda + \frac{c}{T^\beta}$. We are left with optimizing h_λ: $\inf h_\lambda(T) = h_\lambda(T_\lambda) = \frac{c(\beta+1)}{T_\lambda^\beta}$ with $T_\lambda^{\beta+1} = \frac{c\beta}{\operatorname{Re}\lambda}$. If $T_\lambda \leqslant T_0$ then we choose $T = T_\lambda$ and obtain $h_\lambda(T) = a(\operatorname{Re}\lambda)^\alpha$ with $\alpha = \frac{\beta}{\beta+1}$ and $a = c^{\frac{1}{\beta+1}}\frac{\beta+1}{\beta^\alpha}$. Otherwise $T_0\operatorname{Re}\lambda \leqslant \frac{c\beta}{T_0^\beta}$ then we choose $T = T_0$ and obtain $h_\lambda(T) \leqslant \frac{c(\beta+1)}{T_0^\beta}$. □

As a corollary, we state that the resolvent condition with exponential factor

$$\|v\|^2 \leqslant a_0 e^{2a(\operatorname{Re}_+\lambda)^\alpha}\left(\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \in \mathbb{C}, \tag{22}$$

with $\alpha \in (0, 1]$, a and a_0 positive, is necessary for final-observability.

N.b. $\operatorname{Re}_+\lambda := \max\{\operatorname{Re}\lambda, 0\}$ so that (22) for $\lambda > 0$ is nothing but (10).

Theorem 2.4. *If admissibility* (1) *and final-observability* (2) *hold for some T then* (22) *holds with power $\alpha = 1$ and rate $a = T$. If* (2) *holds moreover for all $T \in (0, T_0]$ with the control cost $\kappa_T = c_0 e^{\frac{2c}{T^\beta}}$ for some positive β, c and c_0 then* (22) *holds with power $\alpha = \frac{\beta}{\beta+1} < 1$ and rate $a = c^{\frac{1}{\beta+1}}\frac{\beta+1}{\beta^\alpha}$.*

Proof. By remark 1.1, without loss of generality we may assume that the spectrum of A is contained in a positive half-space $\{z \in \mathbb{C} \mid \operatorname{Re} z \geqslant \lambda_0 > 0\}$. Since

$$\operatorname{Re}\lambda < \lambda_0 \quad \Rightarrow \quad \|v\| \leqslant \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}\|(A-\lambda)v\| \leqslant \frac{1}{|\operatorname{Re}\lambda - \lambda_0|}\|(A-\lambda)v\|, \tag{23}$$

the two implications result from those in proposition 2.3: (21) implies (22) with a greater a_0, and similarly (20) implies (22) with $\alpha = 1$, $a = T$ and a greater a_0. □

Remark 2.5. The fact that final-observability in some time T (2) implies an observability resolvent condition (24) for some unknown positive functions m and M was observed independently by Hans Zwart in [OWR11]. In proposition 2.3, such m and M are given explicitly in (19). The proof was already outlined in remark 1.13 of [Mil08].

Remark 2.6. Some known exponentially localized eigenfunctions allow to prove that a resolvent condition with a better factor (i.e. smaller as $\lambda \to +\infty$) than the exponential one in (22) cannot be necessary for the final-observability for all $T > 0$. For completeness, we write the simplest example explicitly: the eigenfunctions $e_n(x, y, z) = (x + iy)^n$, $n \in \mathbb{N}^*$, of the Laplacian on the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ satisfy $(A - \lambda_n)e_n = 0$ and $\|e_n\| \geqslant ae^{a\sqrt{\lambda_n}}\|Ce_n\|$ for some $a > 0$, where $\mathcal{E} = L^2(S^2)$, $A = -\Delta$ and C is the multiplication by the characteristic function of the complement of any neighborhood of the great circle $\{z = 0\}$ (cf. [Mil08, §4.2.2] for a similar computation), although final-observability (2) holds for any time (cf. [LR95]). Moreover, [Mil08] gives an example (the Laplacian with sextic potential observed from some cone in $\mathbb{R}^3$) where there are eigenfunctions satisfying $\|e_n\| \geqslant ae^{a\lambda_n^{2/3}}\|Ce_n\|$ although final-observability (2) holds for any time.

3. Sufficient conditions for a normal generator

In addition to the framework of §1.1, we assume in this section that $-A$ generates a strongly continuous normal semigroup. Equivalently, A is a normal operator on $\mathcal{E}$ (i.e. A is closed, densely defined and $AA^* = A^*A$) with spectrum contained in a left half-space (i.e. there exists $\gamma \in \mathbb{R}$ such that $\lambda \in \sigma(A)$ implies $\operatorname{Re}\lambda \leqslant \gamma$). The reason for this new assumption, is that such a normal operator A has a spectral decomposition E (a.k.a. projection valued measure) which commutes with any operator which commutes with A, defines spectral projections $\mathbf{1}_{\operatorname{Re} A<\lambda} = E(\{z \in \sigma(A) \mid \operatorname{Re} z < \lambda\})$, spectral spaces $\mathcal{E}_\lambda = \mathbf{1}_{\operatorname{Re} A<\lambda}\,\mathcal{E}$, and more generally provides a simple functional calculus, cf. e.g. [Rud73].

The main result of this section is theorem 3.7 which gives sufficient resolvent conditions to prove final-observability for all $T > 0$ by the Lebeau-Robbiano strategy under the additional assumption that the semigroup is analytic, cf. §3.3. The Lebeau-Robbiano strategy in [Mil10a] for normal semigroups is recalled in §3.2 together with a new logarithmic variant proved in greater generality in §6. We want to stress the similarity between the usual sufficient condition for the Lebeau-Robbiano strategy and some necessary and sufficient condition given in [RTTT05] for the validity of the resolvent condition (5). Therefore we begin in §3.1 by generalizing this so-called "wavepacket condition" of [RTTT05] to normal semigroups and to resolvent conditions with non-constant coefficients.

3.1. **Wavepackets condition.**

We generalize the wavepacket condition introduced in [CFNS91, RTTT05] for A selfadjoint with compact resolvent. Indeed the key result of [RTTT05] is that (25) for D and d constant is equivalent to (24) for M and m constant.

Proposition 3.1. *The observability resolvent condition*

$$\|v\|^2 \leqslant M(\lambda)\|(A - \lambda)v\|^2 + m(\lambda)\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda > 0, \tag{24}$$

implies the wavepackets condition

$$\|v\|^2 \leqslant d(\lambda)\|Cv\|^2, \quad v \in \mathbf{1}_{|A-\lambda|^2 \leqslant D(\lambda)}\,\mathcal{E}, \quad \lambda > 0, \tag{25}$$

for any function $d > m$ with $D = \frac{1-\frac{m}{d}}{M}$ (e.g. $d = 2m$ and $D = \frac{1}{2M}$).

The wavepackets condition (25) *and the admissibility resolvent condition* (19) *imply the observability resolvent condition* (24) *for any function* $m > d$ *with* $M = \delta L + \frac{1+\delta l}{D}$*, where* $\delta = \left(\frac{1}{d} - \frac{1}{m}\right)^{-1}$ *(e.g.* $m = 2d$ *and* $M = 2dL + \frac{1+2dl}{D}$*).*

Proof. Let $v \in \mathbf{1}_{|A-\lambda|^2 \leqslant D(\lambda)} \mathcal{E}$. By the spectral theorem $\|(A-\lambda)v\|^2 \leqslant D(\lambda)\|v\|^2$. Plugging this in (24) yields (25) with $d(\lambda) = \frac{m}{1-DM}$ since $1 - DM = \frac{m}{d} > 0$.

To prove the converse, we introduce the projection $v_\lambda = \mathbf{1}_{|A-\lambda|^2 \leqslant D(\lambda)} v$ of $v \in \mathrm{D}(A)$, and $v_\lambda^\perp = v - v_\lambda$. Using $\|Cv_\lambda\|^2 \leqslant (1+\varepsilon^2)\|Cv\|^2 + (1+\varepsilon^{-2})\|Cv_\lambda^\perp\|^2$, $\varepsilon(\lambda) > 0$, and applying (19) to estimate this last term, then plugging this in (25) yields

$$\|v\|^2 \leqslant d(1+\varepsilon^2)\|Cv\|^2 + d(1+\varepsilon^{-2})L\|(A-\lambda)v_\lambda^\perp\|^2 + (1 + dl(1+\varepsilon^{-2}))\|v_\lambda^\perp\|^2.$$

But the spectral theorem implies $\|v_\lambda^\perp\|^2 \leqslant \frac{1}{D}\|(A-\lambda)v_\lambda^\perp\|^2$, so that (24) holds with $m = d(1+\varepsilon^2)$ and $M = (1+\varepsilon^{-2})dL + \frac{1+dl(1+\varepsilon^{-2})}{D}$. □

Remark 3.2. For example, the following resolvent condition

$$\|v\|^2 \leqslant \frac{M}{\lambda}\|(A-\lambda)v\|^2 + m(\lambda)\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda > 0,$$

(which is equivalent to the observability of the wave equation associated to A, when m is constant and admissibility holds), implies the wave packet condition

$$\|v\|^2 \leqslant 2m(\lambda)\|Cv\|^2, \quad v \in \mathbf{1}_{|A-\lambda| \leqslant \sqrt{\frac{\lambda}{2M}}} \mathcal{E}, \quad \lambda > 0.$$

3.2. **The Lebeau-Robbiano strategy for normal semigroups.** The observation operator $C \in \mathcal{L}(\mathrm{D}(A), \mathcal{F})$ is said to satisfy the *observability condition on spectral subspaces* $\mathcal{E}_\lambda = \mathbf{1}_{\operatorname{Re} A < \lambda} \mathcal{E}$ with exponent $\alpha \in (0,1)$ and rate $a > 0$ if there exists positive a_0 and λ_0 such that

$$\|v\|^2 \leqslant a_0 e^{2a\lambda^\alpha}\|Cv\|^2, \quad v \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0. \tag{26}$$

This condition is the starting point of the Lebeau-Robbiano strategy. In the original framework of [LR95] it writes (14) as recalled in §1.4. In such cases where A is selfadjoint with compact resolvent it is a *condition on sums of eigenfunctions*. The more general framework of [Mil10a] calls it *observability on growth subspaces*. N.b. this condition can be considered as another kind of wavepackets condition: (26) can be written as (25) with the spectral projection $\mathcal{E}_\lambda$ of $\mathcal{E}$ on the left half-plane with abscissa lower than λ replacing the spectral projection of $\mathcal{E}$ on the ball of radius $\sqrt{D(\lambda)}$ and center on the real axis with abscissa λ.

We only recall the simpler version of the main result in [Mil10a] in the current framework of a normal semigroup (cf. [Mil10a, §3.6]) with the simplest estimate of the cost κ_T (n.b. the reference operator C_0 is the identity, hence does not appear).

Theorem 3.3. *Assume the admissibility condition* (1)*, or there exists* $\omega \in \mathbb{R}$ *and* $\theta \in [0, \frac{\pi}{2})$ *such that the spectrum of* A *satisfies* $\sigma(A) \subset \{z \in \mathbb{C} \mid \arg(z-\omega) \leqslant \theta\}$.

If the observability condition on spectral subspaces in time

$$\|e^{-TA}v\|^2 \leqslant a_0 e^{2a\lambda^\alpha + \frac{2b}{T^\beta}} \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}_\lambda, \ T \in (0, T_0), \ \lambda \geqslant \lambda_0, \tag{27}$$

holds with β*,* λ_0*,* a_0*,* a*,* b *positive and* $\alpha = \frac{\beta}{\beta+1}$*, then final-observability* (2) *holds for all* $T > 0$ *with the control cost estimate* $\limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$.

If the observability condition on spectral subspaces (26) *holds then* (27) *holds for all* $b > 0$ *and the resulting estimate is:* $\limsup_{T \to 0} T^\beta \ln \kappa_T \leqslant 2a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}$.

N.b. the time-dependent condition (27) is [Mil10a, (10)]. It generalizes the condition in [LR95, §2, proposition 1] used in the original strategy before the time-independent condition (14) was introduced, cf. also [Léa10, §4].

Remark 3.4. Admissibility here can even be replaced by the weak time smoothing effect (introduced in [Mil10a, lemma 3.1] generalizing [TT10]), with the β of (27):

$$\forall x \in \mathcal{E},\ \forall t > 0,\ e^{-tA}x \in \mathrm{D}(A), \quad \text{and} \quad \limsup_{t\to 0} t^\beta \ln\|Ae^{-tA}\| = 0.$$

If $-A$ generates an analytic semigroup then this is satisfied for any $\beta > 0$ (cf. e.g. [EN00, theorem II.4.6]). Since A is normal, analyticity is equivalent to the condition on $\sigma(A)$ stated in theorem 3.3 (cf. e.g. [EN00, corollary II.4.7]). In particular it is satisfied if A is positive self-adjoint (which is the original assumption in [TT10]).

The following is the analogous simpler statement of the variant of the direct Lebeau-Robbiano strategy proved in theorem 6.1:

Theorem 3.5. *Assume the admissibility condition* (1).
If the logarithmic observability condition on spectral subspaces in time

$$\|e^{-TA}v\|^2 \leqslant a e^{\frac{2a\lambda}{(\log\lambda)^\gamma T^\beta}} \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}_\lambda,\ T \in (0,T_0),\ \lambda \geqslant \lambda_0, \tag{28}$$

holds for some $\lambda_0 > 1$*,* $a > 0$*,* $\beta > 0$ *and* $\gamma > \beta + 1$*, then final-observability* (2) *holds for all* $T > 0$*. Alternatively,* (28) *may be replaced by* (13)*.*

N.b. the assumption (28) is (66) with $\varphi(\lambda) = (\log\lambda)^\gamma$ and $\psi(\omega) = \omega^{\beta+1}$. It is weaker than (27), but theorem 3.5 lacks the control cost estimate of theorem 3.3.

3.3. **Resolvent condition for the Lebeau-Robbiano strategy.**

The following lemma just states a sufficient resolvent condition for an observability condition on spectral subspaces of the same kind as (26). The characterization used in its first sentence can be found in [EN00, corollary II.4.7]. N.b. if A is nonnegative self-adjoint, it applies with $\theta = 0$.

Lemma 3.6. *Assume that the normal semigroup generated by* $-A$ *is bounded analytic, i.e. there exists* $\theta \in [0, \frac{\pi}{2})$ *such that* $\sigma(A) \subset \{z \in \mathbb{C} \mid \arg(z) \leqslant \theta\}$*.*
The observability resolvent condition

$$\|v\|^2 \leqslant \frac{\cos^2\theta}{(\lambda+\lambda_1)^2}\|(A-\lambda)v\|^2 + m(\lambda)\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \lambda_0, \tag{29}$$

with positive λ_1 *and* λ_0 *implies*

$$\|v\|^2 \leqslant \frac{(\lambda_1+\lambda_0)^2}{\lambda_1\lambda_0(\lambda_1+2\lambda_0)}\lambda m(\lambda)\|Cv\|^2, \quad v \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0. \tag{30}$$

Proof. Since $\arg(z) \leqslant \theta$ implies $|\operatorname{Im} z| \leqslant (\tan\theta)\operatorname{Re} z$, we have $|\operatorname{Im} A| \leqslant (\tan\theta)\operatorname{Re} A$. Moreover $0 \leqslant \operatorname{Re} A \leqslant \lambda$ implies $|\operatorname{Re} A - \lambda| \leqslant \lambda$, hence for all $v \in \mathcal{E}_\lambda$, we have $\|(A-\lambda)v\|^2 \leqslant (1+\tan^2\theta)\lambda^2\|v\|^2$. Plugging this in the resolvent condition (29) yields $\left(1 - \frac{\lambda^2}{(\lambda+\lambda_1)^2}\right)\|v\|^2 \leqslant m(\lambda)\|Cv\|^2$. Now $\left(1 - \frac{\lambda^2}{(\lambda+\lambda_1)^2}\right)^{-1} = \frac{\lambda}{\lambda_1}g(\lambda)$ where $g(\lambda) = \frac{(\lambda_1+\lambda)^2}{\lambda(\lambda_1+2\lambda)} = \frac{1}{\mu(1+\mu)}$ is a decreasing function of $\mu = (1+\frac{\lambda_1}{\lambda})^{-1} > 0$ hence a decreasing function of $\lambda > 0$. Using $\lambda \geqslant \lambda_0$ yields $\|v\|^2 \leqslant \frac{\lambda}{\lambda_1}g(\lambda_0)m(\lambda)\|Cv\|^2$ which is (30). ☐

This lemma 3.6 yields the following corollary of the Lebeau-Robbiano strategy in theorems 3.3 and 3.5. N.b. if A is positive self-adjoint, it applies with $\omega_0 = \theta = 0$.

Theorem 3.7. *Assume that the normal semigroup generated by* $-A$ *is analytic, i.e. there exists* $\omega \in \mathbb{R}$ *and* $\theta \in [0, \frac{\pi}{2})$ *such that* $\sigma(A) \subset \{z \in \mathbb{C} \mid \arg(z-\omega) \leqslant \theta\}$*.*

The observability resolvent condition with $\alpha \in (0,1)$, $\omega_0 < \omega$, $\lambda_0 > \omega_0$, and positive a_0 and a,

$$\|v\|^2 \leqslant \frac{\cos^2\theta}{(\lambda-\omega_0)^2}\|(A-\lambda)v\|^2 + a_0 e^{2a\lambda^\alpha}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \lambda_0, \tag{31}$$

implies final-observability (2) *for all time $T > 0$ with the control cost estimate*

$$\limsup_{T\to 0} T^\beta \ln \kappa_T \leqslant 2a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}, \qquad \text{where } \beta = \frac{\alpha}{1-\alpha}.$$

The resolvent condition (31) *with λ^α replaced by $\lambda/\left((\log(\log\lambda))^\alpha \log\lambda\right)$, $\alpha > 2$, and the admissibility condition* (1) *imply final-observability* (2) *for all $T > 0$.*

Proof. Let $A_0 = A - \omega$ and $\lambda_1 = \omega - \omega_0 > 0$. The semigroup generated by $-A_0$ is bounded analytic and normal. The condition (31) implies (29) with A replaced by A_0, with $m(\lambda) = a_0' e^{2a'\lambda^\alpha}$ where $a' > a$ is arbitrary, with $a_0' > 0$ depending on a', and maybe a different λ_0. Therefore we may apply lemma 3.6 to A_0. The resulting (30) implies (26) with a replaced by a'. Hence theorem 3.3 applies to A_0. According to remark 1.1, the resulting cost estimate is still valid for A.

The same proof applies to the second part of theorem 3.7 with theorem 3.3 replaced by theorem 3.5. ☐

Remark 3.8. For exponentially stable normal semigroups (not necessarily analytic), [JZ09, theorem 1.3] proves that the resolvent condition (3) implies final-observability in infinite time (7), which implies final-observability at *some* time T in (2). For exponentially stable normal semigroups which are analytic, theorem 3.7 applies with some $\omega > \omega_0 = 0$ and some θ: it says that the observability resolvent condition with $\alpha \in (0,1)$, λ_0, a_0 and a positive,

$$\|v\|^2 \leqslant \frac{\cos^2\theta}{\lambda^2}\|(A-\lambda)v\|^2 + a_0 e^{2a\lambda^\alpha}\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \lambda_0, \tag{32}$$

implies a stronger conclusion than [JZ09, theorem 1.3]: final-observability for *all* positive times T and a control cost estimate. Comparing (32) with (3), we see that the assumption (32) concerns only large real λ, it is weaker on the second coefficient (it is exponentially increasing instead of polynomially decreasing with λ), but it is stronger on the first coefficient: M is restricted to taking the value $\cos^2\theta$ (e.g. $M = 1$ if A is positive self-adjoint). N.b. concerning this restriction, [GC96] and under weaker assumptions [JZ09, proposition 4.1] prove: if an exponentially stable semigroup satisfies (3) with $M = 1$, then it is exactly-observable in infinite time, which implies final-observability in infinite time (7).

4. Sufficient resolvent condition for a self-adjoint generator

In addition to the framework of §1.1, we assume in this section that A is positive self-adjoint. The main reason is that we shall use the resolvent condition for the exact controllability of the corresponding second order equation $\ddot{w} + Aw = 0$. Since the propagators $\mathrm{Cos}(t) = \cos(t\sqrt{A})$ and $\mathrm{Sin}(t) = (\sqrt{A})^{-1}\sin(t\sqrt{A})$ can be defined for more general operators A (known as generators of a strongly continuous cosine operator functions, cf. [Mil06a]) there is still some hope to extend this section to a more general framework. *N.b. the infimum of the spectrum is denoted* $\inf A$.

4.1. **Preliminaries on second order control systems.**

We introduce the Sobolev scale of spaces based on A. For any $s \in \mathbb{R}$, let H_s denote the Hilbert space $\mathrm{D}(A^{s/2})$ with the norm $\|x\|_s = \|A^{s/2}x\|$ (n.b. $H_0 = \mathcal{E}$).

For simplicity we consider the framework which suits the observability of the wave equation from the interior rather than from the boundary. When C is an interior observation operator, $C \in \mathcal{L}(H_0, \mathcal{F})$ and admissibility is obvious. In this section we make the weaker assumption $C \in \mathcal{L}(H_1, \mathcal{F})$, but this is stronger than the assumption $C \in \mathcal{L}(H_2, \mathcal{F})$ in §1.1. Indeed we consider the second order system with output function y:

$$\ddot{z}(t) + Az(t) = 0, \quad z(0) = z_0 \in H_0, \quad \dot{z}(0) = z_1 \in H_{-1}, \quad y(t) = Cz(t), \tag{33}$$

We rewrite it as a first order system $\dot{w} - i\mathcal{A}w = 0$ in the Hilbert space $\mathcal{W} = H_0 \times H_{-1}$ with norm $\|(z_0, z_1)\|^2 = \|z_0\|_0^2 + \|z_1\|_{-1}^2$. The self-adjoint generator $\mathcal{A}$ is defined by

$$\mathcal{A}(z_0, z_1) = i(-z_1, \tilde{A}z_0) \text{ with domain } \mathrm{D}(\mathcal{A}) = H_1 \times H_0,$$

where $\tilde{A}$ here denotes the extension of A to H_{-1} with domain H_1. The observation operator $\mathcal{C} \in \mathcal{L}(\mathrm{D}(\mathcal{A}), \mathcal{F})$ is defined by $\mathcal{C}(z_0, z_1) = Cz_0$.

The admissibility condition for (33) is (4), i.e.

$$\exists\, \mathrm{Adm}_\tau > 0,\ \forall (z_0, z_1) \in \mathrm{D}(\mathcal{A}), \quad \int_0^\tau \|Cz(t)\|^2 dt \leqslant \mathrm{Adm}_\tau \|(z_0, z_1)\|^2. \tag{34}$$

This condition is equivalent to the resolvent condition, cf. [Mil12, corollary 3.15],

$$\|Cv\|^2 \leqslant L_2(\lambda) \left(\frac{1}{\lambda} \|(A - \lambda)v\|^2 + \|v\|^2 \right), \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \inf A, \tag{35}$$

where the positive function L_2 is (for the time being) constant.

Recall that, if the admissibility condition (34) holds, then exact controllability in time τ is equivalent to exact observability in time τ of (33), i.e.

$$\exists\, \mathrm{Obs}_\tau > 0,\ \forall (z_0, z_1) \in \mathrm{D}(\mathcal{A}), \quad \|(z_0, z_1)\|^2 \leqslant \mathrm{Obs}_\tau \int_0^\tau \|Cz(t)\|^2 dt. \tag{36}$$

As already mentioned in theorem 1.2 of the introduction, the existence of $\tau > 0$ such that the observability condition (36) holds is equivalent to the resolvent condition

$$\|w\|^2 \leqslant M(\lambda) \|(\mathcal{A} - \lambda)w\|^2 + m(\lambda) \|\mathcal{C}w\|^2, \quad w \in \mathrm{D}(\mathcal{A}), \quad |\lambda| \geqslant \inf \sqrt{A}, \tag{37}$$

where the positive functions M and m are (for the time being) constants. Moreover (37) with constant m and M, is equivalent to (cf. [Mil12, corollary 3.18])

$$\|v\|^2 \leqslant M_2(\lambda) \left(\frac{1}{\lambda} \|(A - \lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \inf A, \tag{38}$$

where the positive function M_2 is (for the time being) constant.

We shall need the following more precise statement with variable coefficients L_2, M_2, M and m proved in [Mil12, example 3.17] (e.g. proposition 5.2 gives an example where $L_2(\lambda)$ and $M_2(\lambda)$ increase like $\lambda / \log^2(1 + \lambda)$):

Theorem 4.1. *The resolvent conditions* (35) *and* (38) *with L_2 constant or $L_2(\lambda) \to +\infty$ as $\lambda \to +\infty$, and with $M_2(\lambda) \to +\infty$ as $\lambda \to +\infty$ imply the exact observability condition* (37) *with $m(\lambda) = 8M_2(\lambda^2)$ and $M(\lambda)$ equivalent to some constant times $L_2(\lambda^2)M_2(\lambda^2)$ as $\lambda \to +\infty$. (n.b. $M(\lambda)$ also depends on $\|C\|_{\mathcal{L}(H_1;\mathcal{F})}$).*

We shall also need the fast control cost estimate provided by the control transmutation method, as stated at the end of the proof of [Mil06a, theorem 3.4]:

Theorem 4.2. *There exists $k_* > 0$ such that the admissibility* (34) *and the exact observability* (36) *for some time $\tau > 0$ of the second order system* (33) *imply the final-observability of the first order system* (2) *for all times $T > 0$ with the control cost estimate $\kappa_T \leqslant \mathrm{Obs}_\tau\, k_* \exp(k_* \tau^2 / T)$, $T \in (0, \min\{1, \tau^2\})$.*

4.2. **Main result.** The control transmutation method in [Mil06a] stated in terms of the resolvent conditions in [Mil12] says that the resolvent conditions (35) and (38) when the functions L_2 and M_2 are positive constants imply final-observability (2) for any time $T>0$ with the control cost estimate $\limsup_{T\to 0} T\ln\kappa_T < +\infty$.

Our main result is that, with appropriate admissibility condition, the observability resolvent condition (38) is still sufficient when $M_2(\lambda)$ increases like λ^δ, $\delta\in(0,1)$:

Theorem 4.3. *Assume that the positive self-adjoint operator A and the operator C bounded from $\mathrm{D}(\sqrt{A})$ with the graph norm to $\mathcal{F}$ satisfy the admissibility and observability conditions with nonnegative γ and positive δ, L_* and M_*:*

$$\|Cv\|^2 \leqslant L_*\lambda^\gamma \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|v\|^2\right), \quad v\in \mathrm{D}(A), \quad \lambda \geqslant \inf A, \tag{39}$$

$$\|v\|^2 \leqslant M_*\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v\in \mathrm{D}(A), \quad \lambda \geqslant \inf A. \tag{40}$$

If $\gamma+\delta<1$ then final-observability (2) for the semigroup generated by $-A$ holds for any time $T>0$ with the control cost estimate

$$\limsup_{T\to 0} T^\beta \ln\kappa_T < +\infty, \qquad \textit{where } \beta = \frac{1+\gamma+\delta}{1-\gamma-\delta}.$$

Proof. Setting $L_2(\lambda)=L_*\lambda^\gamma$ and $M_2(\lambda)=M_*\lambda^\delta$, theorem 4.1 yields (37) with $M(\lambda)=m(\lambda)=M'_*\lambda^{2\delta'}$, $\delta'=\gamma+\delta$, and some other positive constant M'_*. Hence,

$$\|w\|^2 \leqslant M'_*\lambda^{2\delta'}\left(\|(\mathcal{A}-\mu)w\|^2+\|\mathcal{C}w\|^2\right), \quad w\in\mathrm{D}(\mathcal{A}), \quad \lambda\geqslant|\mu|\geqslant \inf\sqrt{A}. \tag{41}$$

For any $\lambda>\inf A$, we introduce the restriction $A_\lambda = \mathbf{1}_{A<\lambda} A$ of A to the spectral subspace $\mathcal{E}_\lambda=\mathbf{1}_{A<\lambda}\mathcal{E}$, and similarly the restriction $\mathcal{A}_\lambda=\mathbf{1}_{|\mathcal{A}|<\lambda}\mathcal{A}$ of $\mathcal{A}$ to the spectral subspace $\mathcal{W}_\lambda=\mathbf{1}_{|\mathcal{A}|<\lambda}\mathcal{W}$. N.b. $\mathcal{A}_\lambda$ is associated with A_{λ^2} (rather than A_λ) and $\mathcal{W}_\lambda=\mathcal{E}_{\lambda^2}\times\mathcal{E}_{\lambda^2}$ with the same norm as $\mathcal{W}$ (indeed $\mathcal{A}$ is isomorphic to $\sqrt{A}\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, cf. e.g. [Mil12, theorem 3.8]). Since $\sigma(\mathcal{A}_\lambda)\subset\left\{\mu\in\mathbb{R}\mid\lambda\geqslant|\mu|\geqslant\inf\sqrt{A}\right\}$, applying the last paragraph of theorem 1.2 to the restricted resolvent condition (41) yields the full resolvent condition:

$$\|w\|^2 \leqslant M'_*\lambda^{2\delta'}\left(\|(\mathcal{A}_\lambda-\mu)w\|^2+\|\mathcal{C}w\|^2\right), \quad w\in\mathcal{W}_\lambda, \quad \mu\in\mathbb{R}, \quad \lambda\geqslant\inf\sqrt{A}.$$

By theorem 1.2, this implies that the group generated by $i\mathcal{A}_\lambda$ is exactly observable by $\mathcal{C}$ for all $\tau>\tau_*=\pi\sqrt{M'_*}\lambda^{\delta'}$ with cost $\mathrm{Obs}_\tau\leqslant\frac{2\tau_*^2\tau}{\pi^2(\tau^2-\tau_*^2)}$. Taking $\tau=\sqrt{2}\tau_*$ yields $\mathrm{Obs}_\tau\leqslant 2\pi^{-2}\tau$. By theorem 4.2, this implies final-observability for all times $T>0$ of the semigroup generated by $-A_{\lambda^2}$ with the cost estimate $\kappa_T\leqslant 2\pi^{-2}k_*\tau\exp(k_*\tau^2/T)$, $T\in(0,T_0)$, $T_0\leqslant\min\{1,\tau^2\}$. Since $\lambda\geqslant\inf\sqrt{A}$, we may take $T_0=\min\{1,2\pi^2M'_*(\inf A)^{\delta'}\}$. For notational convenience we change λ^2 into λ. Thus, taking any $c_*>2\pi^2M'_*k_*$, there exists $c_0>0$ such that

$$\|e^{-TA}v\|^2\leqslant c_0e^{c_*\lambda^{\delta'}/T}\int_0^T\|Ce^{-tA}v\|^2dt, \quad v\in\mathcal{E}_\lambda,\ T\in(0,T_0),\ \lambda\geqslant\inf A. \tag{42}$$

Taking α and β such that $\beta=\frac{\alpha}{1-\alpha}$ and $\frac{1}{\beta}+\frac{\delta'}{\alpha}=1$, Young inequality yields

$$\frac{\lambda^{\delta'}}{T}=\frac{(\lambda^\alpha)^{\delta'/\alpha}}{T}\leqslant\frac{\delta'}{\alpha}\lambda^\alpha+\frac{1}{\beta T^\beta}, \qquad \text{with} \quad \alpha=\frac{\delta'+1}{2}, \quad \beta=\frac{1+\delta'}{1-\delta'}.$$

Hence (42) implies (27) for some positive a_0, a and b, with $\alpha\in(1/2,1)$ since $\delta'\in(0,1)$. The Lebeau-Robbiano strategy in theorem 3.3 completes the proof. $\square$

Remark 4.4. The assumption of the control transmutation method corresponds to $\gamma=\delta=0$ in theorem 4.3. The Russell-Weiss condition (3) assumed in the more general result of [JZ09] for normal generators mentioned in remark 3.8 corresponds to $\delta = -1$ in (40). As already mentioned after theorem 1 in §1.3, the condition $\delta < 1$ is sharp when C is bounded.

4.3. **Variants.** Thanks to the improved Lebeau-Robbiano strategy in §6, we replace the polynomial loss $\lambda^{1-\delta}$ in (40) of Theorem 4.3 by a logarithmic loss:

Theorem 4.5. *Assume that the positive self-adjoint operator A and the operator C bounded from $\mathrm{D}(\sqrt{A})$ with the graph norm to $\mathcal{F}$ satisfy the admissibility and observability resolvent conditions with positive α, L_* and M_*:*

$$\|Cv\|^2 \leqslant L_* \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|v\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \inf A, \tag{43}$$

$$\|v\|^2 \leqslant \frac{M_*\lambda}{\log^\alpha(\lambda+1)} \left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathrm{D}(A), \quad \lambda \geqslant \inf A. \tag{44}$$

If $\alpha > 2$ then final-observability (2) *holds for any time $T > 0$.*

Remark 4.6. In Proposition 5.2 below, we give an example satisfying the admissibility condition (1) and (44) with $\alpha = 2$, and such that final-observability (2) does not hold for any time $T > 0$. However this example does not satisfy condition (43).

Proof. The proof is very close to the one of Theorem 4.3, except that we use Theorem 3.5 instead of Theorem 3.3 to conclude. We use the notations $\mathcal{E}_\lambda$, A_λ, $\mathcal{W}$, $\mathcal{A}$, $\mathcal{W}_\lambda$ and $\mathcal{A}_\lambda$ of the proof of Theorem 4.3.

Fix $\lambda \geqslant \inf \sqrt{A}$. By Theorem 4.1 with $L_2(\lambda) = L_*$ and $M_2(\lambda) = \frac{M_*\lambda}{\log^\alpha(\lambda+1)}$, there exists $M'_* > 0$ such that

$$\|w\|^2 \leqslant \frac{M'_*\lambda^2}{\log^\alpha(\lambda+1)} \left(\|(\mathcal{A}-\mu)w\|^2 + \|Cw\|^2\right), \quad w \in D(\mathcal{A}),\ \inf\sqrt{A} \leqslant |\mu| \leqslant \lambda.$$

By the last paragraph of theorem 1.2 applied to the operator $\mathcal{A}_\lambda$,

$$\|w\|^2 \leqslant \frac{M'_*\lambda^2}{\log^\alpha(\lambda+1)} \left(\|(\mathcal{A}_\lambda-\mu)w\|^2 + \|Cw\|^2\right), \quad w \in \mathcal{W}_\lambda,\ \mu \in \mathbb{R}.$$

By Theorem 1.2, the group generated by $i\mathcal{A}_\lambda$ is exactly observable in time $\frac{\lambda\pi\sqrt{2M'_*}}{\log^{\alpha/2}(\lambda+1)}$, with a cost bounded, up to a multiplicative constant, by $\frac{\lambda}{\log^{\alpha/2}(\lambda+1)}$. By Theorem 4.2, this implies final-observability for $e^{-tA_{\lambda^2}}$, in time $T \in (0, T_0)$ (T_0 is independent of λ), with a cost which can be bounded from above by $e^{\frac{a\lambda^2}{T\log^\alpha(\lambda+1)}}$ for some $a > 0$. Hence (for some constant a'),

$$\|e^{-TA}v\|^2 \leqslant a' e^{\frac{a'\lambda}{T\log^\alpha(\lambda+1)}} \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}_\lambda,\ T \in (0,T_0),\ \lambda \geqslant \inf A.$$

and the conclusion of the theorem follows from Theorem 3.5. □

To close this section we give the version of the main result in the framework which suits the observability from the boundary rather than from the interior, i.e. we return to the weaker assumption $C \in \mathcal{L}(H_2, \mathcal{F})$ in §1.1. It uses the definition of Sobolev spaces H_s and norms introduced in §4.1.

Corollary 4.7. *Assume that the positive self-adjoint operator A and the bounded operator $C : \mathrm{D}(A) \to \mathcal{F}$ satisfy the admissibility and observability conditions with nonnegative γ and positive δ, L_* and M_*:*

$$\|Cv\|^2 \leqslant L_*\lambda^\gamma \left(\frac{1}{\lambda}\|(A-\lambda)v\|_1^2 + \|v\|_1^2\right), \quad v \in H_3, \quad \lambda \geqslant \inf A, \tag{45}$$

$$\|v\|_1^2 \leqslant M_*\lambda^\delta \left(\frac{1}{\lambda}\|(A-\lambda)v\|_1^2 + \|Cv\|^2\right), \quad v \in H_3, \quad \lambda \geqslant \inf A. \tag{46}$$

If $\gamma+\delta<1$ then the conclusion of theorem 4.3 still holds.

Proof. It is enough to prove the conclusion (2) for v in the dense space H_3 and with the final-state norm $\|e^{-TA}v\|$ replaced by the larger norm $\|e^{-TA}v\|_1 = \|e^{-TA}\sqrt{A}v\|$ (indeed this is not a stronger conclusion as can be proved using the analyticity of the semigroup for an arbitrary small portion τ of the time T: $\|\tau A e^{-\tau A}\|$ is bounded for $\tau>0$). Equivalently, it is enough to replace C in (2) by $CA^{-1/2}$. To complete the proof of this corollary, we check that theorem 4.3 applies to this new observation operator $CA^{-1/2}$: it is in $\mathcal{L}(\mathrm{D}(\sqrt{A}),\mathcal{F})$ since $C\in\mathcal{L}(\mathrm{D}(A),\mathcal{F})$ and, replacing $v\in H_3$ by $A^{-1/2}v$ with $v\in \mathrm{D}(A)$, (45) and (46) are the needed resolvent conditions. □

5. Not sufficient resolvent conditions: two counterexamples

We first give the concrete example of the quantum harmonic oscillator hamiltonian A on the line, observed from a half line. In this example C is a bounded operator and the resolvent condition (9) of theorem 1 in §1.3 is satisfied in the excluded limit case $\delta=1$ although its conclusion does not hold. N.b. no stronger resolvent condition holds (the precise statement is point (b) in proposition 5.1), the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$ is observable for *some* time T, but the heat semigroup $(e^{-tA})_{t\geqslant 0}$ is not observable for *any* time T.

In the second example, A is a positive self-adjoint operator with compact resolvent on the state space $\mathcal{E}=\ell^2$ of complex (or real) square-summable sequences and C is an observation with one dimensional output space $\mathcal{F}=\mathbb{C}$ (resp. $\mathcal{F}=\mathbb{R}$). In this example C is admissible (but not bounded) and the logarithmic resolvent condition (44) of theorem 4.5 is satisfied in the excluded limit case $\alpha=2$ although its conclusion does not hold. N.b. the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$ is observable for *any* time, but the heat semigroup $(e^{-tA})_{t\geqslant 0}$ is not observable *in a stronger sense* (no sums of eigenfunctions may be driven to zero in finite time).

5.1. **Harmonic oscillator observed from a half line.** Consider $A=-\partial_x^2+V$ on $\mathcal{E}=L^2(\mathbb{R})$ with the quadratic potential $V(x)=x^2$. It is positive self-adjoint with domain $\mathrm{D}(A)=\{u\in H^2(\mathbb{R}) \mid Vu\in L^2(\mathbb{R})\}$. Let $C:\mathcal{E}\to\mathcal{F}=\mathcal{E}$ be the multiplication by the characteristic function of a half line $(-\infty,x_0)$, $x_0\in\mathbb{R}$.

Proposition 5.1. *This harmonic oscillator observed from a half line satisfies:*

(a) *The observation operator C is bounded on $\mathcal{E}$, hence it is admissible for both the heat semigroup $(e^{-tA})_{t\geqslant 0}$ and the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$.*

(b) *The resolvent condition with positive variable coefficients M and m*

$$\|v\|^2 \leqslant M(\lambda)\|(A-\lambda)v\|^2 + m(\lambda)\|Cv\|^2, \quad v\in \mathrm{D}(A), \quad \lambda\in\mathbb{R},$$

holds when M and m are constant functions, but it cannot hold with a function M such that $M(\lambda)\to 0$ as $\lambda\to+\infty$, cf. [Mil10b].

(c) *The Schrödinger group is null-controllable (hence exactly controllable) for any time $T>\pi/2$ (but not for $T<\pi/2$), cf.* [Mil10b].

(d) *The heat semigroup is not null-controllable in any time $T>0$, cf.* [Mil08].

Proof. Point (a) is trivial. For the sake of completeness, we sketch the proof of point (b) in [Mil10b]. The resolvent condition in point (b) writes

$$\int_{-\infty}^{+\infty}|v(x)|^2dx \leqslant M(\lambda)\int_{-\infty}^{+\infty}|-v''(x)+(x^2-\lambda)v(x)|^2dx + m(\lambda)\int_{-\infty}^{x_0}|v(x)|^2dx \quad v\in C_0^\infty(\mathbb{R}), \quad \lambda\in\mathbb{R}. \tag{47}$$

To prove it when M and m are constant functions, we may equivalently restrict it to $\lambda \geqslant 1$ by the last paragraph of theorem 1.2 since the first eigenvalue of A is 1. By the change of variable $u(y) = v(x)$, $y = \sqrt{h}x$, $h = 1/\lambda$, it writes

$$\int_{-\infty}^{+\infty} |u(y)|^2 dy \leqslant \frac{M}{h^2} \int_{-\infty}^{+\infty} |-h^2 u''(y) + (y^2-1)u(y)|^2 dy \\ + m \int_{-\infty}^{\sqrt{h}x_0} |u(y)|^2 dy \quad u \in C_0^\infty(\mathbb{R}), \quad h \in (0,1].$$

Arguing by contradiction, we consider a sequence (h_n) converging to some limit h_∞ in $[0,1]$ and a corresponding sequence of functions (u_{h_n}) such that the left hand side is equal to 1 whereas the right hand side converges to 0. From now on, for brevity, we drop the index n, the variable y and its infinite limits in the integrals. Integrating by parts, $\int |hu_h'|^2 + V|u_h|^2 = \int (-h^2u_h'' + (V-1)u_h)\bar{u_h} + \int |u_h|^2$. This converges to 1 since $\int |u_h|^2 = 1$ and $\int |-h^2u_h'' + (V-1)u_h|^2 \to 0$. Hence $\int |hu_h'|^2$ and $\int V|u_h|^2$ are bounded.

We first consider the case $h_\infty \neq 0$. Since $\left\{u \in L^2(\mathbb{R}) \mid \int |u'|^2 + V|u|^2 < c\right\}$ is compact in $L^2(\mathbb{R})$ for all $c > 0$, extracting a subsequence if needed, we may assume that (u_h) converges to some u in $L^2(\mathbb{R})$. This limit u vanishes on a half-line since $\int_{-\infty}^{\sqrt{h_\infty}x_0} |u|^2 = \lim \int_{-\infty}^{\sqrt{h}x_0} |u_h|^2 = 0$. Taking the weak limit of $-h^2u_h'' + (V-1)u_h$ yields $-h_\infty^2 u'' + (V-1)u = 0$. Hence $u = 0$, contradicting $\int |u|^2 = \lim \int |u_h|^2 = 1$.

In the case $h_\infty = 0$, since (u_h) is bounded in $L^2(\mathbb{R})$, extracting a subsequence if needed, we may assume it has a semiclassical measure μ on the phase space $\mathbb{R} \times \mathbb{R}$ (we refer to [GMMP97] for an introduction to this tool also known as Wigner measure). Since $\int |hu_h'|^2$ and $\int V|u_h|^2$ are bounded, we deduce that, in the terminology of [GMMP97], (u_h) is h-oscillating and compact at infinity (more precisely, $\int_{|y|\geqslant R} |u_h(y)|^2 dy \leqslant \frac{1}{R^2} \int V|u_h|^2$ implies $\limsup_h \int_{|y|\geqslant R} |u_h|^2 \leqslant 1/R^2 \to 0$ as $R \to +\infty$). By [GMMP97, proposition 1.7.ii], this ensures $\mu(\mathbb{R}^2) = \lim_h \int |u_h|^2 = 1$. Since $\int |-h^2u_h'' + (V-1)u_h|^2$ converges to 0, μ is supported on the the circle of radius 1 in $\mathbb{R}^2$ (the characteristic set). Since it converges faster than h^2, μ is invariant by rotation (the hamiltonian flow). Since $\int_{-\infty}^{\sqrt{h}x_0} |u_h|^2 \to 0$, $\mu((-\infty,0) \times \mathbb{R}) = 0$. Combining these last three facts yields $\mu = 0$, contradicting the previous fact $\mu(\mathbb{R}^2) = 1$.

To disprove (47) when $M(\lambda) \to 0$ as $\lambda \to +\infty$ it is sufficient to display an unobserved quasimode in the following sense, keeping the same semiclassical notations: a sequence (u_h) in $L^2(\mathbb{R})$ such that $\int |u_h|^2 \neq 0$ does not depend on h, $\int_{-\infty}^{\sqrt{h}x_0} |u_h|^2 = 0$ for h small enough and $\int |-h^2u_h'' + (V-1)u_h|^2/h^2$ is bounded. We construct (u_h) in the usual WKB form $u_h(x) = a(x)e^{i\varphi(x)/h}$, $h > 0$, where $a \neq 0$ is a smooth amplitude with compact support included in $(0,1)$ and φ is a smooth phase function on $(-1,1)$ satisfying the Hamilton-Jacobi equation $|\varphi'|^2 + V - 1 = 0$, more explicitly $2\varphi(x) = x\sqrt{1-x^2} + \arcsin x$. It fulfils its purpose: $\int |u_h|^2 = \int |a|^2 \neq 0$, $(-\infty, \sqrt{h}x_0] \cap \operatorname{supp}(a) = \emptyset$ for h small enough and $\int |(-h^2u_h'' + (V-1)u_h)/h|^2 \to \int |\varphi''a + 2\varphi'a'|^2 < +\infty$ since

$$-h^2u_h'' + (V-1)u_h = (|\varphi'|^2 + V - 1)u_h - hi(\varphi''a + 2\varphi'a')e^{i\varphi/h} - h^2a''e^{i\varphi/h}.$$

The controllability of the Schrödinger group in some time T results from point (b) and theorem 1.2. Point (c) gives the optimal value of this T obtained in [Mil10b] by space-time semiclassical measures. We shall not recall this lengthier proof here.

For the sake of completeness, we recall the proof of point (d) in [Mil08, §4.3.1]. We have to disprove the observability condition (2) which we rewrite:

$$\exists \kappa_T > 0, \forall v \in L^2(\mathbb{R}), \quad \int_{-\infty}^{+\infty} |e^{-TA}v|^2(x)\,dx \leqslant \kappa_T \int_0^T \int_{-\infty}^{x_0} |e^{-tA}v|^2(x)\,dx\,dt. \tag{48}$$

The Schwartz distribution kernel on $\mathbb{R}^2$ of the operator e^{-tA} is called the Hermite kernel and denoted $(x, y) \mapsto e^{-tA}(x, y)$ (if the initial state is the Dirac mass at y then $x \mapsto e^{-tA}(x, y)$ is the state at time t). The key idea is to consider the initial condition $v(x) = e^{-t_0 A}(x, y)$ in (48) for given $T > 0$, $t_0 > 0$ and $y > 0$, and let $y \to +\infty$ in the end. By the semigroup property, $(e^{-tA}v)(x) = e^{-(t+t_0)A}(x, y)$. The first Hermite function $\phi_0(x) = \pi^{-1/4}e^{-|x|^2/2}$ is the normalized eigenfunction of $A = -\Delta + |x|^2$ corresponding to the lowest eigenvalue $\lambda_0 = 1$. Let $T_0 = t_0 + T$. Writing the semigroup in a Hilbert basis of eigenfunctions yields

$$\exists C_0 > 0, \forall y \in \mathbb{R}^d, \quad \int_{-\infty}^{+\infty} |e^{-TA}v|^2(x)\,dx \geqslant e^{-2T_0\lambda_0}|\phi_0(y)|^2 = C_0 e^{-|y|^2}. \tag{49}$$

Mehler's explicit formula for the Hermite kernel is (cf. e.g. [Dav89, prop 4.3.1]):

$$e^{-tA}(x, y) = \frac{e^{-t}}{\sqrt{\pi(1 - e^{-4t})}} \exp\left(-\frac{(1 + e^{-4t})(x^2 + y^2) - 4e^{-2t}xy}{2(1 - e^{-4t})}\right).$$

The function $a(t) = \frac{1+e^{-4t}}{1-e^{-4t}}$ is decreasing for $t > 0$, hence $a(T_0) > \lim_\infty a = 1$ and

$$|e^{-tA}(x, y)|^2 \leqslant \frac{1}{\pi(1 - e^{-4t_0})} \exp\left(-a(T_0)(x^2 + y^2) + \frac{4|x_0|y}{1 - e^{-4t_0}}\right),$$

for all $x < x_0$, $y > 0$ and $t \in (t_0, T_0)$. This implies: $\exists C_1 \in (1, a(T_0))$, $\exists C_2 > 0$,

$$|e^{-tA}(x, y)|^2 \leqslant C_2 e^{-C_2 x^2 - C_1 y^2}, \quad x < x_0,\ y > 0,\ t \in [t_0, T_0].$$

Therefore, setting $C_3 = TC_2 \int_{-\infty}^{x_0} e^{-C_2 x^2}dx$ yields:

$$\exists C_1 > 1, \exists C_3 > 0, \forall y \in \mathbb{R}_+, \quad \int_0^T \int_{-\infty}^{x_0} |e^{-tA}v|^2(x)\,dx\,dt \leqslant C_3 e^{-C_1 y^2}. \tag{50}$$

The combination of (49) and (50) as $y \to +\infty$ proves that the null-controllability inequality (48) does not hold for any T. □

N.b. in this example the eigenvalues $\lambda_n = 2n+1$ of A satisfy the second property stated in theorem A.1, i.e. the divergence of $\sum_{n\geqslant 1} \frac{1}{\lambda_n}$, but Appendix A does not apply because the output space $\mathcal{F} = L^2(\mathbb{R})$ is not one-dimensional. Instead we resorted to Mehler's *explicit formula for the semigroup kernel.*

N.b. the proof of the resolvent condition by contradiction using semiclassical measures follows [Bur02] and [BZ04, theorem 8] where the lower-order coefficients of A are at most bounded. Here the semiclassical reduction takes advantage of the homogeneity of the *unbounded* potential as in [Mil08, Mil10b].

5.2. **The log threshold.**

Proposition 5.2. *There exists a positive self-adjoint operator A on ℓ^2 with dense domain $\mathrm{D}(A)$ and compact resolvent, and an observation operator $C \in \mathcal{L}(\mathrm{D}(A), \mathbb{C})$ with the following properties:*

(a) *The observation C is admissible for the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$ and the heat semigroup $(e^{-tA})_{t\geqslant 0}$.*

(b) *The following logarithmic resolvent observability condition holds for some positive constant M:*

$$\|v\|^2 \leqslant \frac{M}{\log^2(\lambda+1)}\|(A-\lambda)v\|^2 + M\|Cv\|^2, \quad v \in \mathrm{D}(A), \quad \lambda > 0.$$

(c) *For any time $T > 0$, the Schrödinger group is controllable by C in time T.*

(d) *For any time $T > 0$, the heat semigroup is not controllable by C in time T. More precisely, given any nonzero finite sum of eigenfunctions of A as initial state, there is no input steering it to zero at time T.*

N.b. in this example $C \in \mathcal{L}(\mathrm{D}(A^{\varepsilon+1/2}), \mathbb{C})$ for all $\varepsilon > 0$, but $C \notin \mathcal{L}(\mathrm{D}(\sqrt{A}), \mathbb{C})$.

Proof. Let $(e_n)_{n\geqslant 1}$ be the canonical Hilbert basis of ℓ^2. For $x \in \ell^2$, denote by $x_n = (x, e_n)$ its n-th coordinate.

Consider the operator A on ℓ^2 with domain $\mathrm{D}(A)$ defined by

$$\mathrm{D}(A) = \Big\{x \in \ell^2 \mid \sum_{n\geqslant 1} n^2(\log n)^2 x_n^2 < \infty\Big\}, \qquad Ae_n = n\log(n+1)e_n,\ n \geqslant 1.$$

Note that $A = f(B)$, where f is the convex function $t \mapsto t\log(t+1)$, and B is the operator on ℓ^2 with domain $\mathrm{D}(B) = \Big\{x \in \ell^2 \mid \sum_{n\geqslant 1} n^2 x_n^2 < \infty\Big\}$ defined by $Be_n = n\, e_n$, $n \geqslant 1$. Consider the observation operator C defined by

$$Cx = \sum_{n\geqslant 1} x_n.$$

Note that $C \in \mathcal{L}(\mathrm{D}(B), \mathbb{C}) \subset \mathcal{L}(\mathrm{D}(A), \mathbb{C})$. Indeed $C \in \mathcal{L}(\mathrm{D}(A^{\varepsilon+1/2}), \mathbb{C})$ for all $\varepsilon > 0$, but $x_n = 1/(n\log n\log(\log n))$, $n > 1$, proves that $C \notin \mathcal{L}(\mathrm{D}(\sqrt{A}), \mathbb{C})$.

By Parseval identity, C is admissible for the group e^{itB}, and this group is observable in a time π by C. By [Mil12, Theorems 2.3 and 2.4], these two facts imply the following resolvent conditions for some positive constants L_0, M_0:

$$\|Cx\|^2 \leqslant L_0\|(B-\lambda)x\|^2 + L_0\|x\|^2, \quad \lambda > 0 \tag{51}$$

$$\|x\|^2 \leqslant M_0\|(B-\lambda)x\|^2 + M_0\|Cx\|^2, \quad \lambda > 0. \tag{52}$$

Let us show that (51) and (52) imply (for some positive constants L, M)

$$\|Cx\|^2 \leqslant \frac{L}{\log^2(\lambda+1)}\|(A-\lambda)x\|^2 + L\|x\|^2, \quad \lambda > 0 \tag{53}$$

$$\|x\|^2 \leqslant \frac{M}{\log^2(\lambda+1)}\|(A-\lambda)x\|^2 + M\|Cx\|^2, \quad \lambda > 0. \tag{54}$$

This follows from [Mil12, Theorem 3.2] as in [Mil12, Example 3.4]. We sketch the proof for the sake of completeness. Fixing $\mu > 0$, we consider the function $g_\mu : t \mapsto \frac{f(t)-f(\mu)}{t-\mu}$ $(g_\mu(\mu) = f'(\mu))$. As f is convex, g_μ is nondecreasing on $(0, +\infty)$. By functional calculus, using that B is positive, we get

$$g_\mu(0)\|(B-\mu)x\| \leqslant \|g_\mu(B)(B-\mu)x\|,$$

which yields

$$\log(\mu+1)\|(B-\mu)x\| \leqslant \|(A-f(\mu))x\|.$$

Plugging $\mu = f^{-1}(\lambda)$ here yields that (51) and (52) imply (53) and (54).

The point (b) of the proposition is exactly (54). It implies point (c) by theorem 1.2 since the resolvent of A is compact (cf. [RTTT05, proposition 6.6.4] or [Mil12, corollary 2.14]).

Inequality (53) implies the admissibility of C for the group e^{itA} by [Mil12, Theorem 2.3]. To complete the proof of point (a) of the proposition, we compute the admissibility of C for the heat semigroup $(e^{-tA})_{t\geqslant 0}$ (setting $\lambda_k = k\log(k+1) \leqslant k$ and ending with Hilbert's inequality, cf. [Hil06, HLP88]), for $x \in \mathrm{D}(A)$,

$$\int_0^T \|Ce^{-tA}x\|^2 dt \leqslant \int_0^{+\infty} \|Ce^{-tA}x\|^2 dt \leqslant \int_0^{+\infty} \sum_{j,k\geqslant 1} e^{-t(\lambda_j+\lambda_k)}|x_j||x_k|dt$$
$$\leqslant \sum_{j,k\geqslant 1} \frac{|x_j||x_k|}{\lambda_j+\lambda_k} \leqslant \sum_{j,k\geqslant 1} \frac{|x_j||x_k|}{j+k} \leqslant \pi \sum_{j\geqslant 1} |x_j|^2.$$

Point (d) results from the divergence of $\sum_{n\geqslant 1} \frac{1}{n\log(n+1)}$ and AppendixA. $\square$

Remark 5.3. In [Mil12, Example 3.13], this diagonal operator A on ℓ^2 is interpreted as a function of the Dirichlet Laplacian on $(0,1)$, and C as a boundary observation.

6. The direct Lebeau-Robbiano strategy without explicit cost

This section concerns the so-called “Lebeau-Robbiano strategy” for proving final-observability (2) already mentioned in §3.2. It originates in the heat control strategy in [LR95]. In order to estimate how the control cost blows up as the time available to perform it tends to zero, it was recently revisited in [Mil10a], which gives more background and references. The main result of this section, theorem 6.1, improves [Mil10a] when estimating the control cost is not a goal. Its simpler statement in the specific normal semigroups framework of §3.2 can be found in theorem 5 of §1.3.

Throughout this section we essentially use the more general framework of [Mil10a] in order to encompass all the applications discussed there. It is recalled in §6.1 in the form suitable for the improvement. The general statement of the new result, the discussion of logarithmic conditions and their application to logarithmic anomalous diffusions are given in §6.2. The key idea and proof of the result are given in §6.3.

6.1. **Framework.** We consider the abstract differential equation:

$$\dot{\phi}(t) = A\phi(t), \quad \phi(0) = x \in \mathcal{E}, \quad t \geqslant 0, \tag{55}$$

where $A : \mathrm{D}(A) \subset \mathcal{E} \to \mathcal{E}$ is the generator of a strongly continuous semigroup $(e^{tA})_{t\geqslant 0}$ on a Hilbert space $\mathcal{E}$. The solution is $\phi(t) = e^{tA}x$.

N.b. here the generator is A whereas it was $-A$ in previous sections.

We also consider an observation operator C continuous from $\mathrm{D}(A)$ with the graph norm to another Hilbert space $\mathcal{F}$ (norms in $\mathcal{E}$ and $\mathcal{F}$ are both denoted $\|\cdot\|$). For simplicity we assume that C satisfies the *admissibility* condition

$$\int_0^T \|Ce^{tA}x\|^2 dt \leqslant \mathrm{K}_T\|x\|^2, \quad x \in \mathrm{D}(A), \quad T > 0. \tag{56}$$

This assumption could probably be replaced by some time smoothing effect as in theorem 3.3, cf. [Mil10a, lemma 3.1].

The goal is to prove that for all $T > 0$ there is a cost $\kappa_T > 0$ such that final-observability in time T holds, i.e.

$$\|e^{TA}x\|^2 \leqslant \kappa_T \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathrm{D}(A). \tag{57}$$

We now generalize the three conditions of the Lebeau-Robbiano strategy in [Mil10a] in order to encompass our logarithmic improvement.

Let T_0 and λ_0 be positive constants. Let φ and ψ be two positive increasing continuous functions defined on $(\lambda_0, +\infty)$ and $(1/T_0, +\infty)$ respectively such that φ, $\lambda \mapsto \lambda/\varphi(\lambda)$ and $\omega \mapsto \psi(\omega)/\omega$ tend to $+\infty$ at $+\infty$. E.g. ψ satisfies these assumptions if $\omega \mapsto \psi(\omega)/\omega$ is a positive increasing continuous function on $(1/T_0, +\infty)$ tending to $+\infty$ at $+\infty$. N.b. the assumptions in [Mil10a] correspond to $\varphi(\lambda) = \lambda^{1-\alpha}/a$, $\alpha \in (0,1)$, $a > 0$, and $\psi(\omega) = b\omega^{\beta+1}$, $\beta > 0$, $b > 0$.

We assume that there is a nondecreasing family of semigroup invariant spaces $\mathcal{E}_\lambda \subset \mathcal{E}$, $\lambda \geqslant \lambda_0$ (i.e. $e^{tA}\mathcal{E}_\lambda \subset \mathcal{E}_\lambda \subset \mathcal{E}_{\lambda'}$, $t > 0$, $\lambda' > \lambda$) satisfying the semigroup growth property (namely some time-decay): there exists $m_0 > 0$ and $m \geqslant 0$,

$$\|e^{tA}x\| \leqslant m_0 e^{m\lambda/\varphi(\lambda)} e^{-\lambda t}\|x\|, \quad x \perp \mathcal{E}_\lambda, \quad t \in (0, T_0), \quad \lambda \geqslant \lambda_0. \tag{58}$$

When A is a normal operator (as in §3.2) these *growth spaces* $\mathcal{E}_\lambda$ are naturally defined by the functional calculus as the spectral subspaces $\mathcal{E}_\lambda = \mathbf{1}_{\mathrm{Re}\, A > -\lambda}\,\mathcal{E}$ which satisfy (58) with $m_0 = 1$ and $m = 0$ by the spectral theorem. Note the interpretation of λ as a spectral abscissa in this simple case.

We also assume that there is an observation operator $C_0 \in \mathcal{L}(\mathrm{D}(A), \mathcal{F})$ which satisfies the final-observability property: there exists $b_0 > 0$ such that

$$\|e^{TA}x\|^2 \leqslant b_0 e^{2T\psi(1/T)} \int_0^T \|C_0 e^{tA}x\|^2 dt, \quad x \in \mathrm{D}(A), \quad T \in (0, T_0). \tag{59}$$

In the original framework of [LR95] and in §3.2, the *reference operator* C_0 is the identity operator which always satisfies (59) with $b_0 = \sup_{t\in[0,T_0]} \|e^{-tA}\|^2$ and $\psi(\omega) = \frac{1}{2}\omega \log \omega$, all the more with $\psi(\omega) = b\omega^{\beta+1}$, $\beta > 0$, $b > 0$. Some applications require C_0 to be a non-trivial projection as discussed in [Mil10a, §3.7].

Finally, generalizing the "condition on sums of eigenfunctions" (26), we assume the *observability on the growth subspaces* $\mathcal{E}_\lambda$ *relative to the reference operator* C_0: there exists $a_0 > 0$ such that there are positive constants a_0 and a such that

$$\|C_0 x\|^2 \leqslant a_0 e^{2\lambda/\varphi(\lambda)} \|Cx\|^2, \quad x \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0. \tag{60}$$

6.2. The direct Lebeau-Robbiano strategy and logarithmic conditions.

The statement of the "direct Lebeau-Robbiano strategy without explicit cost" is:

Theorem 6.1. *In the framework of §6.1, assuming in particular admissibility* (56) *and relative observability on growth subspaces* (58)*,* (59) *and* (60)*, if*

$$s \mapsto \frac{1}{\psi^{-1}\left(\frac{\varphi(q^s)}{p}\right)} \text{ is integrable at } +\infty \text{ for some } p > m+1 \text{ and } q > 1, \tag{61}$$

then final-observability (57) *holds for all* $T > 0$.

N.b. instead of assuming (59) *and* (60) *separately, we may as well assume* (66)*.*

N.b. if (61) *holds for some* $q > 1$ *then it holds for all* $q > 1$*.*

This theorem is proved in the next §6.3. Here we discuss some applications.

As explained after (59), $\psi(\omega) = \omega^{\beta+1}$, $\beta > 0$, and $\psi(\omega) = \omega \log \omega$ are interesting examples of ψ. Here are some admissible functions φ for these ψ.

Lemma 6.2. *The integrability condition* (61) *holds for all* $p > 0$ *and* $q > 1$ *with the following functions* (ψ, φ) *(hence also with* $(b\psi, \varphi/a)$ *for all* $a > 0$ *and* $b > 0$*):*

$$\begin{aligned} &\psi(\omega) = \omega^{\beta+1},\ \beta > 0, &&\text{and} \quad \varphi(\lambda) = (\log \lambda)^\alpha,\ \alpha > \beta + 1, \\ \text{or} \quad &\psi(\omega) = \omega \log \omega &&\text{and} \quad \varphi(\lambda) = (\log(\log \lambda))^\alpha \log \lambda,\ \alpha > 2. \end{aligned}$$

Proof. This follows from straightforward computations. With the substitution $\varphi(q^s) = p\psi(\omega)$, (61) is equivalent to the integrability of ds/ω at $+\infty$. We only give details for the latter case. In this case this substitution writes

$$s(\log q) \log^\alpha(s \log q) = p\omega \log \omega. \tag{62}$$

Taking the logarithm of (62) yields $\log s \sim \log \omega$ as $s \to +\infty$. Since $(x \log^\alpha x)' \sim \log^\alpha x$ as $x \to +\infty$, taking the derivative of (62) yields $\frac{ds}{d\omega} \sim \frac{p \log \omega}{\log q \log^\alpha s}$ as $s \to +\infty$. Hence (61) is equivalent to the integrability of $d\omega/(\omega \log^{\alpha-1} \omega)$ at $+\infty$. This well-know Betrand integral is convergent if and only if $\alpha > 2$. □

Now we state the corollary corresponding to the original framework of [LR95]. Let M be a smooth connected compact d-dimensional Riemannian manifold with metric g and boundary $\partial M \neq \emptyset$. Let Δ denote the Laplace-Beltrami operator on $L^2(M)$ with domain $\mathrm{D}(\Delta) = H_0^1(M) \cap H^2(M)$ defined by g. N.b. the results are already interesting when (M, g) is a smooth connected bounded domain of the Euclidean space $\mathbb{R}^d$, so that $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$, as stated in theorem 7.

In this application, the state and input spaces are $\mathcal{E} = \mathcal{F} = L^2(M)$, the growth spaces are the spectral spaces defined after (58), i.e. $\mathcal{E}_\lambda$ is the linear span of the eigenfunctions of $-\Delta$ with eigenvalues lower than λ. The reference operator C_0 is the identity operator and the observation operator C is the multiplication by the characteristic function χ_Ω of an open subset $\Omega \neq \emptyset$ of M, i.e. it truncates the input function outside the control region Ω.

As recalled in (14) of §1.4, $A = -\Delta$ satisfies (26) with exponent $\alpha = 1/2$. Hence $A = -\sqrt{-\Delta}$ satisfies (26) with $\alpha = 1$. We deduce that (60) holds for $A = -B\varphi(B)$, $B = \sqrt{-\Delta}$, with some possibly greater a_0 and a, when

$$\begin{aligned} &\varphi(\lambda) = (\log(1 + \log(1 + \lambda)))^\alpha \log(1 + \lambda), \quad \alpha > 2, \\ \text{or} \quad &\varphi(\lambda) = (\log(1 + \lambda))^\alpha, \quad \alpha > 1. \end{aligned} \tag{63}$$

The main step of this straightforward computation is: since $\varphi(\lambda) \ll \lambda$, $\mu = \lambda/\varphi(\lambda)$ implies $\log \mu \sim \log \lambda$ and $\lambda = \mu\varphi(\lambda) \sim \mu\varphi(\mu)$. Applying theorem 6.1 with lemma 6.2 yields:

Theorem 6.3. *The anomalous diffusion on the compact manifold M with Dirichlet boundary conditions defined by $A = -\sqrt{-\Delta}\varphi(\sqrt{-\Delta})$ and (63), with input u:*

$$\partial_t \phi - A\phi = \chi_\Omega u, \ \phi(0) = \phi_0 \in L^2(M), \ u \in L^2([0, T] \times M),$$

is null-controllable from any non-empty open subset Ω of M in any time $T > 0$.

Remark 6.4. Theorem 6.3 for $\varphi(\lambda) = \lambda^{1/\beta}$, $\beta > 0$, with the control cost estimate $\limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$ is [Mil10a, theorem 4.1]. It is still an open problem whether theorem 6.3 holds for $\varphi(\lambda) = 1$, although theorem A.2 rather indicates that it does not. This problem is the null-controllability for $\partial_t \phi + \sqrt{-\Delta}\phi = \chi_\Omega u$. Unique continuation holds since $(\partial_t + \sqrt{-\Delta})\phi = 0 \Rightarrow (\partial_t^2 + \Delta)\phi = 0$. We mention some available "transmutation formulas" that could be relevant to this problem:

$$e^{-t\sqrt{-\Delta}} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{t}{t^2 + s^2} \cos(s\sqrt{-\Delta}) ds = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \frac{t}{s^{3/2}} e^{-t^2/(4s)} e^{s\Delta} ds.$$

6.3. **Proof of theorem 6.1.** To bound the cost κ_T obtained by the Lebeau-Robbiano strategy, the crucial lemma [Mil10a, lemma 2.1] partitions the time interval $(0, T]$ into an infinity of intervals with lengths in geometric progression. The key idea in this section is to consider instead a geometric progression of the "spectral abscissa" λ. This means that the dependence of the resulting cost κ_T on T is no longer explicit. In other words, the following alternative lemma exploited in this section does not bound the cost κ_T but allows more general partitions of $(0, T]$:

Lemma 6.5. *Let $\lambda_1 > 0$ and $q > 1$. Consider a positive decreasing continuous function τ on $(\lambda_1, +\infty)$ such that $\tau(\lambda) \to 0$ as $\lambda \to +\infty$ and satisfying*

$$s \mapsto \tau(q^s) \text{ is integrable at } +\infty. \tag{64}$$

Also consider a positive function f on $(\lambda_1, +\infty)$ such that $f(\lambda) \to 0$ as $\lambda \to +\infty$.

The approximate observability estimate

$$f(\lambda)\|e^{\tau(\lambda)A}x\|^2 - f(q\lambda)\|x\|^2 \leqslant \int_0^{\tau(\lambda)} \|Ce^{tA}x\|^2 dt, \quad x \in \mathrm{D}(A), \quad \lambda \geqslant \lambda_1, \tag{65}$$

implies final-observability (57) *for all* $T > 0$.

Proof. Define the geometric sequence $\lambda_{k+1} = q\lambda_k$, $k \in \mathbb{N}^*$, and the corresponding sequence of time lapses $\tau_k = \tau(\lambda_k)$. Due to the integrability condition (64) and the monotonicity of τ, the series $\sum_k \tau_k$ converges. Hence $T_n = \sum_{k\geqslant n} \tau_k$ defines a decreasing sequence of times converging to zero. Applying (65) to $x = e^{T_{k+1}A}y$ and $\lambda = \lambda_k$ yields

$$f(\lambda_k)\|e^{T_k A}y\|^2 - f(\lambda_{k+1})\|e^{T_{k+1}A}y\|^2 \leqslant \int_{T_{k+1}}^{T_k} \|Ce^{tA}y\|^2 dt, \quad y \in \mathrm{D}(A), \quad k \geqslant 1.$$

Since the left hand side is a telescoping series, adding these inequalities yields

$$f(\lambda_N)\|e^{T_N A}y\|^2 - f(\lambda_k)\|e^{T_k A}y\|^2 \leqslant \int_{T_k}^{T_N} \|Ce^{tA}y\|^2 dt, \quad y \in \mathrm{D}(A), \quad k \geqslant N \geqslant 1.$$

Taking the limit $k \to \infty$ yields (57) with $T = T_N$ and $\kappa_T = 1/f(\lambda_N)$ since $f(\lambda_k)$ converges to zero and the continuous function $t \mapsto \|e^{tA}y\|$ is bounded on the compact set $[0, T_N]$. This completes the proof of the lemma since for all $T > 0$ there exists N such that $T_N < T$. □

We proceed with the proof of theorem 6.1. For ease of exposition, we start with the case $m = 0$ in (58) and complete the general case at the very end of §6.3. Plugging (60) in (59) yields

$$\|e^{TA}y\|^2 \leqslant a_0 b_0 e^{2(\lambda/\varphi(\lambda)+T\psi(1/T))} \int_0^T \|Ce^{tA}y\|^2 dt, \quad y \in \mathcal{E}_\lambda,\ T \in (0, T_0),\ \lambda \geqslant \lambda_0.$$

Since $\lambda \mapsto \lambda/\varphi(\lambda)$ and $\omega \mapsto \psi(\omega)/\omega$ tend to $+\infty$ at $+\infty$, taking λ_0 and $1/T_0$ greater if needed, we may assume that these functions are greater than 2 so that their sum is lower than their product. We deduce

$$\|e^{TA}y\|^2 \leqslant a_0 b_0 e^{2T\psi(\frac{1}{T})\frac{\lambda}{\varphi(\lambda)}} \int_0^T \|Ce^{tA}y\|^2 dt,\ y \in \mathcal{E}_\lambda,\ T \in (0, T_0),\ \lambda \geqslant \lambda_0. \tag{66}$$

For a given $x \in \mathrm{D}(A)$, $\lambda \geqslant \lambda_0$ and $\tau \in (0, T_0)$ we introduce an observation time $T = \varepsilon\tau$ with $\varepsilon \in (0, 1)$, the orthogonal projection of x on $\mathcal{E}_\lambda$ denoted x_λ, and $x_\lambda^\perp = x - x_\lambda$.

Since $\mathcal{E}_\lambda$ is semigroup invariant, we may apply (66) to $y = e^{(1-\varepsilon)\tau A}x_\lambda$ and obtain:

$$\|e^{\tau A}x_\lambda\|^2 \leqslant \frac{1}{4g(\tau,\lambda)} \int_{(1-\varepsilon)\tau}^{\tau} \|Ce^{tA}x_\lambda\|^2 dt, \quad g(\tau,\lambda) = \frac{1}{4a_0 b_0} e^{-2T\psi(\frac{1}{T})\frac{\lambda}{\varphi(\lambda)}}. \tag{67}$$

We put the factor 4 in the definition of g because we shall use twice the inequality:

$$\|y + z\|^2 \leqslant 2(\|y\|^2 + \|z\|^2), \quad y \in \mathcal{E}, \quad z \in \mathcal{E}. \tag{68}$$

Using (68) then (56) yields

$$\int_{(1-\varepsilon)\tau}^{\tau} \|Ce^{tA}x_\lambda\|^2 dt \leqslant 2\int_{(1-\varepsilon)\tau}^{\tau} \|Ce^{tA}x\|^2 dt + 2\,\mathrm{K}_{\varepsilon\tau}\|e^{(1-\varepsilon)\tau A}x_\lambda^\perp\|^2. \tag{69}$$

Using (68) again, then (67) and finally (69) yields

$$g(\tau,\lambda)\|e^{\tau A}x\|^2 \leqslant \int_{(1-\varepsilon)\tau}^{\tau} \|Ce^{tA}x\|^2 dt + \mathrm{K}_{\varepsilon\tau}\|e^{(1-\varepsilon)\tau A}x_\lambda^\perp\|^2 + 2g(\tau,\lambda)\|e^{\tau A}x_\lambda^\perp\|^2.$$

Applying (58) with $m = 0$ to $x_\lambda^\perp$ yields

$$g(\tau,\lambda)\|e^{\tau A}x\|^2 - m_0^2\left(\mathrm{K}_{\varepsilon\tau}\, e^{-2(1-\varepsilon)\tau\lambda} + 2g(\tau,\lambda)e^{-2\tau\lambda}\right)\|x_\lambda^\perp\|^2 \leqslant \int_0^\tau \|Ce^{tA}x\|^2 dt.$$

Since $\|x_\lambda^\perp\| \leqslant \|x\|$,$\mathrm{K}_{\varepsilon\tau} \leqslant \mathrm{K}_{T_0}$ and $g(\tau,\lambda) \leqslant \frac{1}{4a_0b_0}$, we set $g_0 = m_0^2(\mathrm{K}_{T_0} + \frac{1}{2a_0b_0})$ and deduce the approximate observability estimate: for all $x \in \mathrm{D}(A)$,

$$g(\tau,\lambda)\|e^{\tau A}x\|^2 - g_0 e^{-2(1-\varepsilon)\tau\lambda}\|x\|^2 \leqslant \int_0^\tau \|Ce^{tA}x\|^2 dt, \quad \lambda \geqslant \lambda_0,\ \tau \in (0, T_0). \tag{70}$$

As noted after (61), we may assume without loss of generality that it holds for some $p > q > 1$. In order to apply lemma 6.5, define the functions τ and f by

$$\frac{1}{\varepsilon\tau(\lambda)} = \psi^{-1}\left(\frac{\varphi(\lambda)}{r}\right),\ r = \frac{q}{1-\varepsilon},\ f(\lambda) = g(\tau(\lambda),\lambda) = \frac{1}{4a_0b_0}e^{-2\varepsilon(1-\varepsilon)\tau(\lambda)\lambda/q}.$$

These functions are well-defined on $(\lambda_1, +\infty)$ for λ_1 large enough. Taking ε small enough ensures $r \in (q, p)$. Since $r < p$ and ψ^{-1} is increasing, the integrability condition (61) also holds with p replaced by r, hence (64) is satisfied. The assumptions on φ and ψ in §6.1 ensure that τ decreases, $\tau(\lambda) \to 0$ as $\lambda \to +\infty$, and $g(\tau,\lambda) \to 0$ as $(\tau,\lambda) \to (0,+\infty)$. Therefore $f(\lambda) \to 0$ as $\lambda \to +\infty$. Equivalently $\tau(\lambda)\lambda \to \infty$ as $\lambda \to +\infty$. Using $\tau(q\lambda) \leqslant \tau(\lambda)$, this yields for λ_1 large enough

$$f(q\lambda) \geqslant \frac{1}{4a_0b_0}e^{-2\varepsilon(1-\varepsilon)\tau(\lambda)\lambda} \geqslant g_0 e^{-2(1-\varepsilon)\tau(\lambda)\lambda}, \quad \lambda \geqslant \lambda_1.$$

Therefore (70) implies (65), and lemma 6.5 completes the proof in the case $m = 0$.

We now complete the general case $m \neq 0$ in (58). The proof uses (58) only once: in the equation before (70). We may divide this equation by $e^{2m\lambda/\varphi(\lambda)}$ and keep the same right hand side since $e^{-2m\lambda/\varphi(\lambda)} \leqslant 1$. This yields (70) with $g(\tau,\lambda)$ replaced by $g(\tau,\lambda)e^{-2m\lambda/\varphi(\lambda)}$. Recalling that $\omega \mapsto \psi(\omega)/\omega$ is greater than 1 (as a result of increasing λ_0), this amounts to replacing φ by $\varphi/(m+1)$ in the definition of g and eventually in the integrability condition (61).

Remark 6.6. We take this opportunity to correct misprints in the proof of [Mil10a, lemma 3.4]: the definition of u is $u(t) = \kappa C e^{(T-t)A}\psi_0$ and, conversely, $f(T) = \int_0^T e^{tA^*}Bu(T-t)dt + e^{TA^*}f_0$ yields $\langle f_0, e^{TA}x\rangle = -\int_0^T \langle u(T-t), Ce^{tA}x\rangle dt + \langle f(T), x\rangle$.

Appendix A. Lack of controllability based on Müntz theorem

For the sake of completeness, we repeat [Mil06b, Appendix] used in §5.2 and take this opportunity to correct some misprints. This appendix concerns control systems having a Riesz basis of eigenvectors and a one-dimensional input space. Theorem A.2 gives a sufficient condition in terms of eigenvalues for a property which is much stronger than the lack of null-controllability: no sum of eigenvectors can be steered to zero. It is based on the following generalized Müntz theorem recalled from [Red77, theorem 7]:

Theorem A.1. *Let $\{\zeta_n\}_{n\in\mathbb{N}}$ be a sequence of distinct non zero complex numbers and let $\{e_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of exponential functions defined by $e_n(t) = \exp(\zeta_n t)$. If $\{\zeta_n\}_{n\in\mathbb{N}}$ satisfies one of these properties:*

i) $\exists\varepsilon > 0$, $\displaystyle\sum_n \frac{1}{|\zeta_n|^{1+\varepsilon}} = \infty$,

ii) $\displaystyle\sum_n |\mathrm{Re}\,\frac{1}{\zeta_n}| = \infty$,

iii) $\{|\zeta_n|\}_{n\in\mathbb{N}}$ *increases and there exists a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers such that* $\displaystyle\sum_n \frac{1}{n^{\theta_n}} < \infty$, *and* $\displaystyle\sum_n \frac{1}{|\zeta_n|^{\theta_n}} = \infty$, *then, for all $T>0$, $\{e_n\}_{n\in\mathbb{N}}$ is complete in $L^2(0,T;\mathbb{C})$, i.e. the only vector orthogonal to this set is 0 (equivalently, any function of $L^2(0,T;\mathbb{C})$ can be approximated in the norm of this space by linear combinations of these exponential functions).*

On a Hilbert space $\mathcal{X}$ we consider the system described by the following differential equation for $t\geqslant 0$:

$$\dot{x}(t)+Ax(t)=bu(t), x(0)=x^0\in\mathcal{X},\ u\in L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{C})\ . \tag{71}$$

We assume that $-A$ is the infinitesimal generator of a C_0-semigroup $\{e^{-tA}\}_{t\geqslant 0}$ on $\mathcal{X}$, which has a sequence of normalized eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ forming a Riesz basis of $\mathcal{X}$, with associated eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, that is, $A\phi_n=\lambda_n\phi_n$. We denote by $\mathcal{X}_1$ the Hilbert space obtained by choosing the graph norm on the domain $D(A)$ of the unbounded operator A on $\mathcal{X}$, by $\mathcal{X}_{-1}$ the space dual to $\mathcal{X}_1$, and we keep the same notation for the extension of $\{e^{-tA}\}_{t\geqslant 0}$ to a semigroup on $\mathcal{X}_{-1}$. We also assume that the "control vector" b is in $\mathcal{X}_{-1}$ so that the solution $x\in C(0,T;\mathcal{X}_{-1})$ of (71) is defined for $T\geqslant 0$ by the integral formula:

$$x(T)=e^{-TA}x^0+\int_0^T e^{-(T-t)A}bu(t)dt\ . \tag{72}$$

There is a sequence of eigenvectors $\{\psi_n\}_{n\in\mathbb{N}}$ of A^* forming a Riesz basis of $\mathcal{X}$, with associated eigenvalues $\{\bar{\lambda}_n\}_{n\in\mathbb{N}}$, which is bi-orthogonal to $\{\phi_n\}_{n\in\mathbb{N}}$, i.e. $\langle\phi_n,\psi_n\rangle=1$ and $\langle\phi_n,\psi_m\rangle=0$ if $m\neq n$. We introduce the coefficients $b_n=\langle b,\psi_n\rangle$ in the expansion $b=\sum_{n\in\mathbb{N}}b_n\phi_n$.

Theorem A.2. *Assume that $b_n\neq 0$ for all n larger than some integer N_b. If the set of distinct non zero eigenvalues of A satisfies one of the properties stated in theorem A.1, then, for all non zero initial state x^0 which is a finite linear combination of the eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ and for all $T>0$, there is no input function $u\in L^2(0,T;\mathbb{C})$ such that the solution $x\in C(0,T;\mathcal{X}_{-1})$ of (71) satisfies $x(T)=0$.*

Proof. Introducing the coefficients $x_n(t)=\langle x(t),\psi_n\rangle$, (72) writes $x_n(T)=e^{-\lambda_nT}x^0_n+\int_0^T e^{-\lambda_n(T-t)}b_nu(t)dt$. With the notation $y_n(t)=\exp(\lambda_nt)$, $x(T)=0$ writes:

$$\forall n\in\mathbb{N},\quad -x^0_n=b_n\int_0^T y_n(t)u(t)dt\ . \tag{73}$$

We make the assumptions on $\{b_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of the theorem. Arguing by contradiction, we also assume that there are $T>0$, $x^0\neq 0$ which is a finite linear combination of the $\{\phi_n\}_{n\in\mathbb{N}}$, and $u\in L^2(0,T;\mathbb{C})$ such that (73) holds. Let x^0_N be the nonzero coefficient of x^0 with the greatest index, i.e. $x^0_N\neq 0$ and $x^0_n=0$ for $n>N$. Let $M=\max\{N_b,N\}$. For all $n>M$, on the one hand $M\geqslant N_b$ implies $b_n\neq 0$, on the other hand $M\geqslant N$ implies $x^0_n=0$, so that (73) implies $\int_0^T y_n(t)u(t)dt=0$. The set of distinct non zero values of $\{\lambda_n\}_{n>M}$ also satisfies the same property stated in theorem A.1 as $\{\lambda_n\}_{n\in\mathbb{N}}$, so that the corresponding subset of $\{y_n\}_{n>M}$ is complete in $L^2(0,T;\mathbb{C})$, and therefore $u=0$. Plugging this in (73) with $n=N$ yields the contradiction: $0\neq -x^0_N=b_N\int_0^T y_n(t)u(t)dt=0$. □

Acknowledgments

The second author was partially supported by the "Agence Nationale de la Recherche" (ANR), Project CISIFS, grant ANR-09-BLAN-0213-02. He wishes to thank Institut Henri Poincaré (Paris, France) for providing a very stimulating environment during the "Control of Partial and Differential Equations and Applications" program in the Fall 2010. This paper was announced in [OWR11].

References

[Bur02] N. Burq, *Lower bounds for shape resonances widths of long range Schrödinger operators*, Amer. J. Math. **124** (2002), no. 4, 677–735.

[BZ04] N. Burq and M. Zworski, *Geometric control in the presence of a black box*, J. Amer. Math. Soc. **17** (2004), no. 2, 443–471.

[CFNS91] G. Chen, S. A. Fulling, F. J. Narcowich, and S. Sun, *Exponential decay of energy of evolution equations with locally distributed damping*, SIAM J. Appl. Math. **51** (1991), no. 1, 266–301.

[Dav89] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

[EN00] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.

[Erv09] S. Ervedoza, *Spectral conditions for admissibility and observability of wave systems: applications to finite element schemes*, Numer. Math. **113** (2009), no. 3, 377–415.

[Erv11] ———, *Spectral conditions for admissibility and observability for Schrödinger systems: Applications to finite element approximation schemes*, Asymptot. Anal. **71** (2011), no. 1–2, 1–32.

[EZZ08] S. Ervedoza, C. Zheng, and E. Zuazua, *On the observability of time-discrete conservative linear systems*, J. Funct. Anal. **254** (2008), no. 12, 3037–3078.

[GC96] P. Grabowski and F. M. Callier, *Admissible observation operators. Semigroup criteria of admissibility*, Integral Equations Operator Theory **25** (1996), no. 2, 182–198.

[GMMP97] P. Gérard, P. A. Markowich, N. J. Mauser, and F. Poupaud, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math. **50** (1997), no. 4, 323–379.

[Hil06] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Vierte Mitteilung.*, Gött. Nachr. **1906** (1906), 157–227 (German).

[HLP88] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition.

[JL99] D. Jerison and G. Lebeau, *Nodal sets of sums of eigenfunctions*, Harmonic analysis and partial differential equations (Chicago, IL, 1996), Univ. Chicago Press, 1999, pp. 223–239.

[JP06] B. Jacob and J. R. Partington, *On controllability of diagonal systems with one-dimensional input space*, Systems Control Lett. **55** (2006), no. 4, 321–328.

[JPP07] B. Jacob, J. R. Partington, and S. Pott, *Interpolation by vector-valued analytic functions, with applications to controllability*, J. Funct. Anal. **252** (2007), no. 2, 517–549.

[JPP09] ———, *Zero-class admissibility of observation operators*, Systems Control Lett. **58** (2009), no. 6, 406–412.

[JS07] B. Jacob and R. Schnaubelt, *Observability of polynomially stable systems*, Systems Control Lett. **56** (2007), no. 4, 277–284.

[JZ04] B. Jacob and H. Zwart, *Counterexamples concerning observation operators for C_0-semigroups*, SIAM J. Control Optim. **43** (2004), no. 1, 137–153.

[JZ09] ———, *On the Hautus test for exponentially stable C_0-groups*, SIAM J. Control Optim. **48** (2009), no. 3, 1275–1288.

[Léa10] M. Léautaud, *Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems*, J. Funct. Anal. **258** (2010), no. 8, 2739–2778.

[Liu97] K. Liu, *Locally distributed control and damping for the conservative systems*, SIAM J. Control Optim. **35** (1997), no. 5, 1574–1590.

[LR95] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), no. 1-2, 335–356.

[LRL09] J. Le Rousseau and G. Lebeau, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM Control Optim. Calc. Var. (2009), to appear.

[LZ98] G. Lebeau and E. Zuazua, *Null-controllability of a system of linear thermoelasticity*, Arch. Rational Mech. Anal. **141** (1998), no. 4, 297–329.

[Mil05] L. Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*, J. Funct. Anal. **218** (2005), no. 2, 425–444.

[Mil06a] ———, *The control transmutation method and the cost of fast controls*, SIAM J. Control Optim. **45** (2006), no. 2, 762–772.

[Mil06b] ———, *On the controllability of anomalous diffusions generated by the fractional Laplacian*, Math. Control Signals Systems **18** (2006), no. 3, 260–271.

[Mil08] ———, *Unique continuation estimates for sums of semiclassical eigenfunctions and null-controllability from cones*, http://hal.archives-ouvertes.fr/hal-00411840, 2008.

[Mil10a] ———, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Contin. Dyn. Syst. B **14** (2010), no. 4, 1465–1485.

[Mil10b] ———, *Geometric conditions and sharp finite time for exact internal controllability in quantum wells*, manuscript, 2010.

[Mil12] ———, *Resolvent conditions for the control of unitary groups and their approximations*, J. of Spectr. Theory **2** (2012), no. 1, 1–55.

[MZ06] S. Micu and E. Zuazua, *On the controllability of a fractional order parabolic equation*, SIAM J. Control Optim. **44** (2006), no. 6, 1950–1972.

[OWR11] *Wellposedness and controllability of evolution equations*, Oberwolfach Rep. **7** (2011), no. 4, 3269–3309, Abstracts from the workshop held December 12–18, 2010, organized by Birgit Jacob, Jonathan Partington, Sandra Pott, Hans Zwart, Report No. 56/2010.

[Phu01] K.-D. Phung, *Observability and control of Schrödinger equations*, SIAM J. Control Optim. **40** (2001), no. 1, 211–230.

[Phu02] ———, *Null controllability of the heat equation as singular limit of the exact controllability of dissipative wave equation under the Bardos-Lebeau-Rauch geometric control condition*, Comput. Math. Appl. **44** (2002), no. 10-11, 1289–1296.

[Red77] R. M. Redheffer, *Completeness of sets of complex exponentials*, Advances in Math. **24** (1977), no. 1, 1–62.

[RTTT05] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak, *A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator*, J. Funct. Anal. **226** (2005), no. 1, 193–229.

[Rud73] W. Rudin, *Functional analysis*, McGraw-Hill Book Co., New York, 1973.

[RW94] D. L. Russell and G. Weiss, *A general necessary condition for exact observability*, SIAM J. Control Optim. **32** (1994), no. 1, 1–23.

[Sei05] T. I. Seidman, *On uniform null controllability and blowup estimates*, Control theory of partial differential equations, Lect. Notes Pure Appl. Math., vol. 242, Chapman & Hall/CRC, 2005, pp. 213–227.

[Sei08] ———, *How violent are fast controls. III*, J. Math. Anal. Appl. **339** (2008), no. 1, 461–468.

[TT10] G. Tenenbaum and M. Tucsnak, *On the null-controllability of diffusion equations*, ESAIM Control Optim. Calc. Var. (2010).

[TW09] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser Verlag, Basel, 2009.

[XLY08] Gen Qi Xu, Chao Liu, and Siu Pang Yung, *Necessary conditions for the exact observability of systems on Hilbert spaces*, Systems Control Lett. **57** (2008), no. 3, 222–227.

[ZY97] Q. Zhou and M. Yamamoto, *Hautus condition on the exact controllability of conservative systems*, Internat. J. Control **67** (1997), no. 3, 371–379.